Bachelor thesis

A curious connection between
2-dimensional topological quantum
field theories and commutative Frobenius algebras

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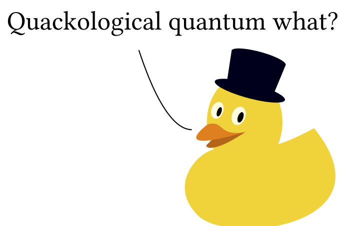

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1 Introduction

Overview

The cobordism category $\mathbf{nCob^{or}}$ consists of manifolds as its objects and oriented cobordisms between these manifolds as its morphisms. It has a monoidal product, namely the disjoint union, and a natural symmetric structure. The same is true for the category of vector spaces $\mathbf{Vect}_{\Bbbk}$ over a field $\Bbbk$ equipped with the usual tensor product. Such categories are called symmetric monoidal categories and functors respecting these structures are called symmetric monoidal functors. In particular, a symmetric monoidal functor from $\mathbf{nCob^{or}}$ into $\mathbf{Vect}_{\Bbbk}$ is called an n-dimensional topological quantum field theory, or TQFT for short.

There is a special kind of an algebra called a Frobenius algebra and there is a connection between these algebras and topology. This connection allows us to understand the commutative Frobenius algebras by understanding $\mathbf{2Cob^{or}}$ and 2-dimensional TQFTs. This will be the main theme of this text.

The well known classification theorems for manifolds of dimensional one and two allows us to completely understand the 2-dimensional cobordism category $\mathbf{2Cob^{or}}$ which in turn gives us a handful of relations there. We will also discover that the defining relations for a Frobenius algebra, when presented in a graphical way, are of a topological nature. Since the symmetric monoidal functors preserves these relations, we will see that the image of the circle under a 2-dimensional TQFT is in fact a commutative Frobenius algebra in $\mathbf{Vect}_{\Bbbk}$.

From there we move on to the main result (corollary 7.2) in this text, which is an equivalence of categories, namely the category of two-dimensionals TQFTs and the category of commutative Frobenius algebras. It other words, we are going to prove that

$$\mathbf{2TQFT}^{\mathbf{or}}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk}.$$

We will mainly be following the book [Koc04] by Joachim Kock.

Why so many pages?

The reader should not be put off by the relatively high number of pages. The reason why there are so many pages is because this text contains a large number of illustrations and diagrams to make the arguments more clear. One can jump directly to section 4, 5 and 6 if one want to go straight to the main theme of this text and use the earlier sections as reference when needed.

Acknowledgements

I would like to express my deepest gratitude and appreciation to my supervisor Dr Claudia Scheimbauer for all the guidance and help I have received. I also wish to thank my family and my girlfriend for all the support they have given me.

Conventions

1. We read cobordisms from left to right unless otherwise is stated. That means that in-boundaries are on the left and out-boundaries are on the right.

2. Given two morphisms $f : X \to Y$ and $g : Y \to Z$ we write the composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ as $g \circ f : X \to Z$ and sometimes even just as gf. We write $f(x)$ for the value of f at x. Note that both these notations differs from what is done in [Koc04].

List of symbols

$\overline{M}$	M with reversed orientation
$\mathrm{Hom}_{\mathscr{C}}(X, Y)$	The set of morphisms from X to Y in the category $\mathscr{C}$
∂M	The boundary of M (as a manifold)
$\partial_{\mathrm{in}} M$	The in-boundary of M
$\partial_{\mathrm{out}} M$	The out-boundary of M
$A \amalg B$	Disjoint union of A and B
$A \longrightarrow B$	An arrow from A to B
$B \subset A$	B is a subset of A
$M : A \Rightarrow B$	A cobordism (class) M from A to B
$V \oplus W$	Direct sum of V and W
$V \otimes W$	Tensor product of V and W
$V \times W$	Cartesian product of V and W
V^*	The dual space of V
$X \in \mathscr{C}$	X is an object in the category $\mathscr{C}$

2 Manifolds

We will work with abstract manifolds, so we do not assume our manifolds to be embedded in any ambient Euclidean space. We will mainly be interested in smooth and oriented manifolds as these will be the objects and morphisms of the cobordism category.

2.1 Definitions and examples

2.1.1 Smooth manifolds

The most elementary manifold is the topological manifold which is intuitively understood to be any space which locally around any point looks like Euclidean space.

Definition 2.1. An *n-dimensional topological manifold* M is a second countable Hausdorff space which is locally homeomorphic to Euclidean n-space. In other words: for every point $p \in M$, there exists an open subset $U \subset M$ containing p and a homeomorphism

$$\phi : U \to \phi(U) \subset \mathbb{R}^n$$

We call n is the *dimension* of M, and sometimes write "n-manifold" when we mean an n-dimensional manifold.

Definition 2.2. A *chart* (U, ϕ) on a n-manifold M consists of an open subset $U \subset M$ and a homeomorphism $\phi : U \to V$ where V is an open subset of $\mathbb{R}^n$. A family of such charts $\{(U_i, \phi_i)\}_{i\in\Lambda}$ where $\bigcup_{i\in\Lambda} U_i = M$ is called an *atlas* of M.

We call the maps $\phi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(U_j \cap U_i) \to \phi_i(U_j \cap U_i)$ *transition maps*. They are maps between open subsets of Euclidean space.

Definition 2.3. An atlas of a manifold M^n is *smooth* if each of the transition maps are smooth. In other words: every $\phi_{ij} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

We say two atlases of a manifold are *smoothly equivalent* if their union is a smooth atlas. This defines an equivalence relation on the set of atlases on the manifold.

We use this relation and define a *smooth structure* $\mathcal{A}$ on M as an equivalence class of atlases.

Definition 2.4. A *smooth manifold* $(M, \mathcal{A})$ is a manifold M together with a smooth structure $\mathcal{A}$ on M. When we say that M is a smooth manifold, it is understood that it comes with a smooth structure.

Definition 2.5. An atlas A of M is *maximal* if it is not strictly contained in any other atlas of M.

Consider a map f between smooth manifolds M and N. What does it mean for f to be smooth at some point $x \in M$? Using charts we can translate our situation to the familiar Euclidean setting.

Definition 2.6. Let M and N be smooth manifolds and $f : M \to N$ a continuous map. We say that f is *smooth at* $x \in M$ if there exist charts $\phi : U_1 \to V_1$ and $\psi : U_2 \to V_2$ on M and N respectively, where $x \in U_1$ and $f(x) \in U_2$ such that

$$\psi \circ f \circ \phi^{-1} : \phi(f^{-1}(U_2)) \to V_2$$

is smooth.

Definition 2.7. If for all $x \in M$, $f : M \to N$ is smooth at x, we say f is *smooth*. If f is smooth, bijective and has a smooth inverse, then we call f a *diffeomorphism* and we say that M and N are *diffeomorphic*.

2.1.2 Manifolds with boundary

Sticking with our current definition of a (smooth) manifold we would miss a lot of interesting spaces. For example, the cylinder $S^1 \times [0, 1]$ is not a manifold according to our definition, since the points in $S^1 \times \{0, 1\}$ do not have a neighbourhood homeomorphic to an open subset of $\mathbb{R}^2$. The problem obviously has something to do with the edge or boundary, which should motivate our next definition of a manifold with boundary.

We define the *n-dimensional (closed) half space* as

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_n \geq 0\} \subset \mathbb{R}^n$$

and give this the standard subspace topology in Euclidean space.

Definition 2.8. A *n-dimensional manifold with boundary* is a second countable Hausdorff space which is locally homeomorphic to an open subset of $\mathbb{H}^n$.

This also includes our previous definition of a manifold since any open set in $\mathbb{R}^n$ is homeomorphic to an open set in $\mathbb{H}^n$. We let the boundary of $\mathbb{H}^n$ be defined as

$$\partial\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}^n\} \subset \mathbb{H}^n$$

and we call $p \in M$ a *boundary point* of M if there exist a chart $\phi : U \to \mathbb{H}^n$ such that $\phi(p) \in \partial\mathbb{H}^n$. We call this a *boundary chart* and the boundary of M, denoted ∂M, is the collection of all such boundary points. Figure 2.1.1 shows a 2-dimensional manifold with boundary together with a boundary chart.

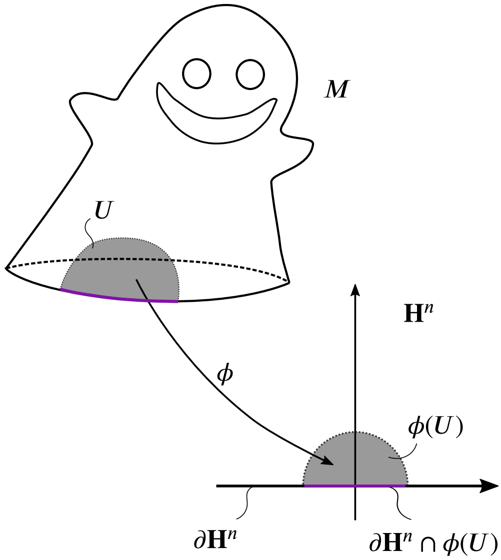


Figure 2.1.1: A ghost-like manifold called "Labolina" with four circles as its boundary.

We now prove a useful lemma about the boundary of a manifold.

Lemma 2.9. The boundary ∂M of an n-dimensional manifold M is an $(n-1)$-dimensional manifold.

Proof. At any point $p \in \partial M$, we have a chart $h : U \xrightarrow{\cong} V \subset \mathbb{H}^n$. Define W to be the set $U \cap \partial M$ (open in ∂M) and consider the restriction of h to W:

$$h\Big|_W : W \xrightarrow{\cong} V \cap \partial\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, 0) \in V\}$$

Observe that $V \cap \partial\mathbb{H}^n$ is the inclusion of an open set of $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$ so it is homeomorphic to an open set of $\mathbb{R}^{n-1}$ by simply "forgetting" the last coordinate which is always zero anyway. This gives us the required chart from ∂M to $\mathbb{R}^{n-1}$ □

Example 2.10.

1. $\mathbb{H}^n$ is a manifold with boundary $\mathbb{R}^{n-1}$.
2. Any n-manifold M is a n-manifold with boundary where $\partial M = \emptyset$.
3. $S^1 \times [0,1]$ is a manifold with boundary $S^1 \sqcup S^1$.
4. A Möbius band is a manifold with boundary homeomorphic to S^1.

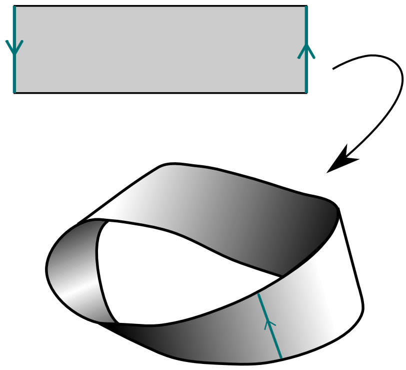


Figure 2.1.2: A Möbius band constructed from a rectangle by twisting and gluing two edges.

2.1.3 Tangent spaces

For a boundaryless ($\partial M = \emptyset$) manifold M embedded in Euclidean space we define the tangent vector space at $x \in M$, denoted $T_x M$, as the image of U under $d_x\phi$ for some parameterization $\phi : U \to M$. It is straightforward to show that this definition is independent of the choice of parameterization. This approach with embedded manifolds is explained in great detail in the first chapter of [GP10].

For abstract smooth manifolds which are not embedded in any ambient space, we would also like to identify a linear space at each point. We will do this by considering all smooth curves trough a given point in our manifold.

Fix a point $x \in M$ and consider the set of all smooth curves γ from an open interval about 0 into M with $\gamma(0) = x$

$$S = \{\gamma : (-\epsilon, \epsilon) \to M \quad | \quad \gamma \text{ smooth and } \gamma(0) = x\}$$

Let ϕ be a chart about x. For $\gamma_1, \gamma_2 \in S$ we define a relation

$$\gamma_1 \sim \gamma_2 \iff d_0(\phi \circ \gamma_1) = d_0(\phi \circ \gamma_2).$$ [1]

That this relation is a equivalence relation follows at once from the properties of equality in $\mathbb{R}^n$. However, we will show that being related is independent of choice of chart. That is, if ϕ and ψ are charts around x, then $\gamma_1 \sim \gamma_2$ under ϕ if and only if $\gamma_1 \sim \gamma_2$ under ψ.

Proof. Let ϕ and ψ be charts around $x \in M$ and let γ_1 and γ_2 be smooth curves into M with $\gamma_1(0) = \gamma_2(0) = x$. Assume $d_0(\phi \circ \gamma_1) = d_0(\phi \circ \gamma_2)$. By the chain rule we have that

$$d_0(\psi \circ \gamma_1) = d_0(\psi \circ \phi^{-1} \circ \phi \circ \gamma_1)$$

[1]For simpler notation we let $d_c(f)$ denote $\frac{df}{dt}(t)\Big|_{t=c}$.

$$= d_{\phi(x)}(\psi\circ\phi^{-1})d_0(\phi\circ\gamma_1)$$
$$= d_{\phi(x)}(\psi\circ\phi^{-1})d_0(\phi\circ\gamma_2)$$
$$= d_0(\psi\circ\gamma_2)$$

(The other way is exactly the same.) Hence the choice of chart does not matter. □

Definition 2.11. Define the tangent vector space of M at $x \in M$, denoted T_xM, to be $S/\sim$. In other words: tangent vectors are the equivalence classes of smooth curves under this relation.

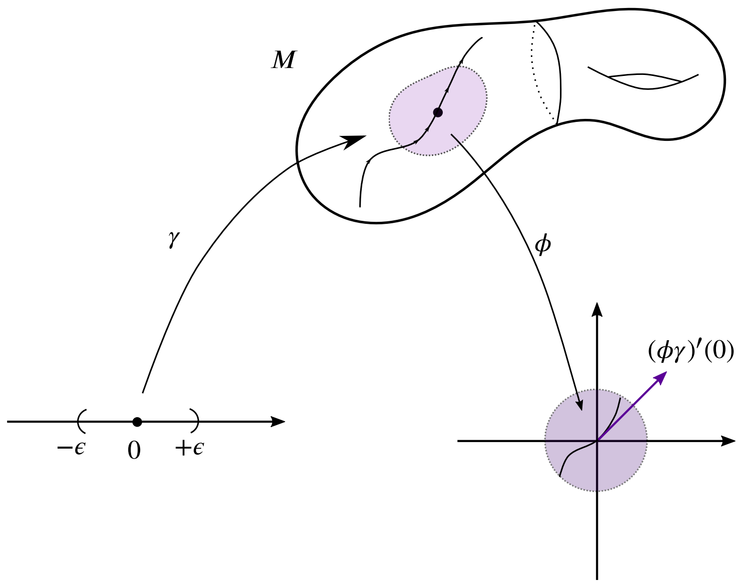


Figure 2.1.3: A smooth curve defines a tangent vector.

A priori, T_xM is nothing more than a set. It might not be obvious, but it turns out that when M is a n-dimensional manifold, T_xM is a n-dimensional vector space. The vector space structure on T_xM is given as follows: Let $[\gamma_1]$ and $[\gamma_2]$ be two classes of smooth curves into M, and let $c \in \mathbb{R}$. Choose a chart ϕ around x. Then we define addition as:

$$[\gamma_1] + [\gamma_2] := [\phi^{-1}(\phi\circ\gamma_1 + \phi\circ\gamma_2)],$$

and scalar multiplication as

$$c \cdot [\gamma_1] := [\phi^{-1}(c \cdot \phi\circ\gamma_1)].$$

Here we made a choice of chart. Of course, one have to show that these definitions are independent of this choice. Details regarding this are given in [HN11, p. 251], including the following lemma:

Lemma 2.12. [HN11, p. 251] The map $T_x M \to \mathbb{R}^n$ given by

$$[\gamma] \longmapsto \frac{d}{dt}(\phi \circ \gamma)(t)\Big|_{t=0}$$

is an isomorphism of vector spaces.

Example 2.13. Let us calculate the tangent vector space of S^1 at $p = (1, 0)$ and show that it is isomorphic to $\mathbb{R}$. We assume S^1 to be embedded in $\mathbb{R}^2$ as in figure 2.1.4 and we use the height function as a chart about p.

$$\phi : S^1 \supset U \to \mathbb{R}$$
$$(x, y) \mapsto y$$

Here U is an open[2] set containing p such that ϕ is a diffeomorphism onto its image. For example we can choose

$$U = \left\{ (\sqrt{1 - y^2}, y) \in \mathbb{R}^2 \text{ such that } |y| < \frac{1}{2} \right\}.$$

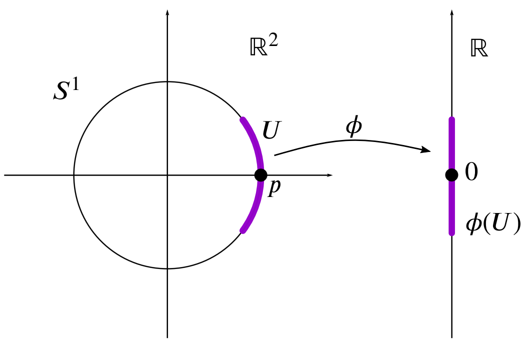


Figure 2.1.4: The circle embedded in $\mathbb{R}^2$.

We construct a family of curves as follows:

$$\gamma_t : \mathbb{R} \to S^1$$
$$\theta \mapsto (\cos t\theta, \sin t\theta)$$

for some $t \in \mathbb{R}$. Observe that $\phi \circ \gamma_t(\theta) = \sin t\theta$ and by the map $D : T_p S^1 \to \mathbb{R}$ in lemma 2.12 we have

$$[\gamma_t] \mapsto \frac{d}{d\theta}(\phi \circ \gamma)(\theta)\Big|_{\theta=0} = t \cos t\theta\Big|_{\theta=0} = t.$$

This shows that D is surjective. Injectivity follows at once from the definition of the equivalence relation. We need to show that D is linear. Using the definition of the vector space structure on $T_p S^1$, we get that

$$c \cdot [\gamma_s] + [\gamma_t] = [\phi^{-1}(c \sin s\theta + \sin t\theta)].$$

[2]We let $S^1 \subset \mathbb{R}^2$ be equipped with the subspace topology.

By straightforward calculation we have that

$$D(c \cdot [\gamma_s] + [\gamma_t]) = cs + t = cD([\gamma_s]) + D([\gamma_t]),$$

proving the linearity of D. We have now explicitly showed that D is in fact an isomorphism, so that $T_pS^1 \cong \mathbb{R}$.

2.1.4 Tangent spaces at boundary points

When dealing with manifolds with boundary, we would like to define the tangent space at points in the boundary. We already know what to do at interior points. To define the tangent space T_pM for a boundary point $p \in \partial M$ we need to consider one-sided derivatives. First, we define the following sets of smooth curves into M:

$$S_p^+ := \{\gamma : [0, 1) \to M \mid \gamma \text{ is smooth and } \gamma(0) = p\},$$

and

$$S_p^- := \{\gamma : (-1, 0] \to M \mid \gamma \text{ is smooth and } \gamma(0) = p\}.$$

Let f be a map from a subset of $\mathbb{R}$ into $\mathbb{H}^n$. Define the *one-sided derivative of f at x* as

$$d^+ f_x = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \quad \left(\text{or } d^- f_x = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}\right).$$

Consider two smooth curves $\gamma_1, \gamma_2 \in S_p^+$ and let (U, ϕ) be a chart about p with $\phi(p) = 0$. We define the relation $\sim^+$ on S_p^+ as follows:

$$\gamma_1 \sim_+ \gamma_2 \iff d^+(\phi \circ \gamma_1)_0 = d^+(\phi \circ \gamma_2)_0,$$

and call an equivalence class under this relation an *inward pointing tangent vector at p*.

Similarly for $\gamma_1, \gamma_2 \in S_p^-$, we define the relation $\sim^-$ on S_p^- as

$$\gamma_1 \sim_- \gamma_2 \iff d^-(\phi \circ \gamma_1)_0 = d^-(\phi \circ \gamma_2)_0,$$

and call an equivalence class under this relation an *outward pointing tangent vector at p*.

Definition 2.14. If $p \in \partial M$, we define the *tangent space at p* denoted T_pM as the union of inward pointing and outward pointing tangent vectors at p.

Note that T_pM is a vector space of dimension $\dim M$, whereas $T_p\partial M$, which is the intersection of inward pointing and outward pointing tangent vectors, is a vector space of dimension $\dim M - 1$ and sits as a subspace inside T_pM.

2.1.5 Orientation

Let V be a real vector space of finite dimension. Given two ordered bases β and β' of V, we define a relation.

$$\beta \sim \beta' \iff \det [I]_{\beta}^{\beta'} > 0$$

where $[I]_{\beta}^{\beta'}$ is the change of basis matrix.

Lemma 2.15. The relation $\sim$ is an equivalence relation on the set B_V of ordered bases of V and it gives exactly two equivalence classes.

Proof.

1. *Reflexivity:* $\beta \sim \beta$ since $[I]_{\beta}^{\beta}$ is the identity matrix.
2. *Symmetry:* Use the fact that $[I]_{\beta'}^{\beta} = \left([I]_{\beta}^{\beta'}\right)^{-1}$. Assume $\det [I]_{\beta}^{\beta'} > 0$, then we have that $\det [I]_{\beta'}^{\beta} = \det \left([I]_{\beta}^{\beta'}\right)^{-1} = \frac{1}{\det [I]_{\beta}^{\beta'}} > 0$. Hence we have that $\beta \sim \beta' \Rightarrow \beta' \sim \beta$.
3. *Transitivity:* Assume $\beta \sim \beta'$ and $\beta' \sim \beta''$. From linear algebra we know that $[I]_{\beta}^{\beta''} = [I]_{\beta'}^{\beta''}[I]_{\beta}^{\beta'}$. It follows from the multiplicative property of the determinant that
$$\det [I]_{\beta}^{\beta''} > 0.$$
Since the change of basis matrix has non-zero determinant, this completes our proof. □

Definition 2.16. An *orientation* on V is a choice of one of the equivalence classes defined by $\sim$. A vector space V together with an orientation is an *oriented* vector space.

Note that choosing a basis β for V completely determines an orientation on V since it represent one of the two orientations.

Remark 2.17. We can encode the same data in the assignment of signs to these two classes. In other words, an orientation on V can be given by a bijection

$$\{-1, +1\} \to {}^{B_V}\!/\!_{\sim}$$

Definition 2.18. Let V and W be vector spaces of the equal dimension. A linear map $T : V \to W$ is said to be *orientation preserving* if it has positive determinant. If the determinant is negative, T is said to be *orientation reversing*.

The standard orientation on $\mathbb{R}^n$ is the class represented by the standard ordered basis $\{e_1, \ldots, e_n\}$. In the special case where $V \cong \mathbb{R}^0$ we have only one basis, namely the empty set, so an orientation on V is then just an assignment of a sign to this one basis. There are numerous ways of defining what an orientation of a manifold is. We will do it as follows:

Definition 2.19. An *oriented atlas* on M is an atlas for which every transition map is orientation preserving. That is, the linear map

$$d_x\phi_{ij} : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$$

is orientation preserving. A manifold M is *orientable* if it admits an oriented atlas and we call a maximal oriented atlas an *orientation* on M. An *oriented* manifold is a manifold together with an orientation. If M is an oriented manifold we denote M with the reversed orientation as $\overline{M}$.

Remark 2.20. This is equivalent to requiring a compatible choice of orientation on each tangent vector space. Note also that if M is an orientable manifold, choosing an orientation on T_xM for a point $x \in M$ completely determines the orientation on the connected component containing x.

Example 2.21. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth map and define the graph of f as

$$\Gamma = \{(x, f(x)) | x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$$

This is a smooth manifold with local parametrization

$$\begin{aligned} \phi : \mathbb{R}^n &\to \Gamma \\ x &\mapsto (x, f(x)) \\ \phi^{-1} : \Gamma &\to \mathbb{R}^n \\ (x, f(x)) &\mapsto x. \end{aligned}$$

We have that

$$\det(d_x\phi_{ij}) = \det(\mathrm{id}) = 1 > 0$$

which proves that Γ is orientable.

Definition 2.22. Let M and N be oriented manifolds (where $\partial M = \emptyset$ or $\partial N = \emptyset$). Then, we define the *induced orientation on the product* $M \times N$ as follows: let $(x, y) \in M \times N$, and let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ be positive bases[3] for T_xM and T_yN, respectively. Define the orientation on $T_{(x,y)}M \times N$ as the one determined by the basis $\{v_1, \dots, v_n, w_1, \dots, w_n\}$.

2.1.6 In- and out-boundaries

An orientation on a manifold with boundary is an orientation of its interior. Normally we make a choice of how this induces orientations on the boundary components. What we want to do is to make a difference between incoming and outgoing boundaries, and this is what we are going to define now. Given an oriented n-manifold M we know that ∂M is a manifold of codimension 1. We let $x \in \partial M$ and observe that $T_x\partial M$ is a subspace of T_xM of codimension 1. We have exactly two unit vectors in T_xM orthogonal to $T_x\partial M$.

[3]By a "positive basis" for T_xM we mean a basis determining the given orientation on M.

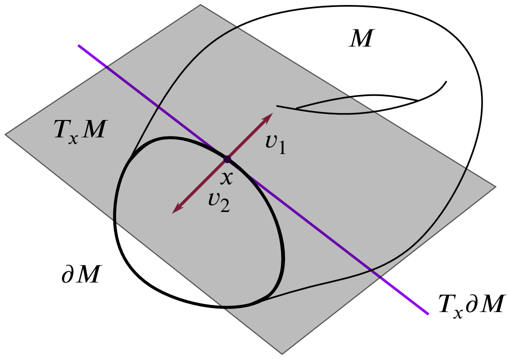


Figure 2.1.5: Normal vectors v_1 and v_2.

We would like to define a notion of incoming and outgoing boundaries. This amounts to choosing orientations of the boundary components relative to the orientation of the (interior of the) manifold. Before we state the definition we take a look at an important example, namely the cylinder $S^1 \times [0,1]$.

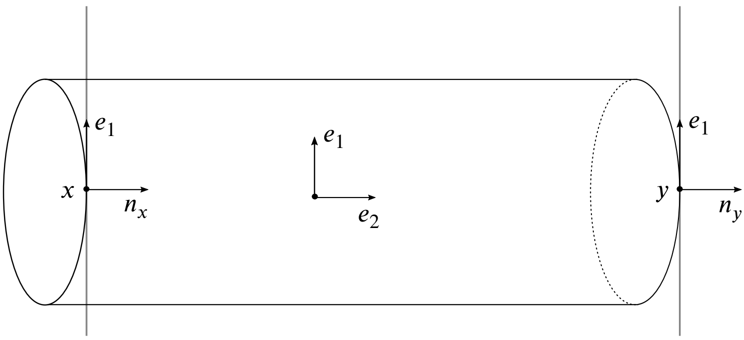


Figure 2.1.6: The unit cylinder.

Say we choose the counter-clockwise orientation on both copies of S^1 (that is the upward pointing vector on the circles). We choose a normal vector so that the basis of T_xS^1 extends to a basis of $T_x(S^1 \times [0,1])$ that gives the desired orientation of $S^1 \times [0,1]$ (the orientation in the middle). This normal vector can either point inward or outward. If it points inwards we say that the boundary component is an in-boundary and an out-boundary if it points outwards. In our example, the circle on the left is an in-boundary and the circle on the right is an out-boundary. We now make this precise in the following definition:

Definition 2.23. Let M be a oriented manifold with boundary and let Σ be a connected component of ∂M. At a point $x \in \Sigma$, let $\{w_1, \dots, w_{n-1}\}$ be a basis of $T_x\Sigma$. We can choose a normal vector n_x in T_xM such that $\{w_1, \dots, w_{n-1}, n_x\}$ determines the chosen orientation on M. If n_x is an inward pointing tangent vector we say that Σ is an *in-boundary*. If n_x is an outward pointing tangent vector, we say that Σ is an *out-boundary*. We denote the collection of all in-boundaries and out-boundaries of M as $\partial_{\text{in}} M$ and $\partial_{\text{out}} M$ respectively.

Note that the boundary of M is the union of in-boundaries and out-boundaries. That is,

$$\partial M = \partial_{\text{in}} M \amalg \partial_{\text{out}} M.$$

We sometimes write *incoming boundary* and *outgoing boundary* when we mean in-boundary and out-boundary, respectively.

Remark 2.24. Note that when we are drawing cobordisms we will always follow the convention that in-boundaries are on the left and out-boundaries are on the right (unless otherwise is stated).

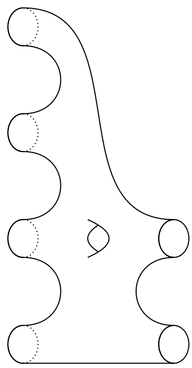


Figure 2.1.7: Oriented cobordism with four circles as its in-boundary and two circles as its out-boundary.

2.2 Classification of low dimensional oriented manifolds

A simple classification of compact manifolds is known for dimensions 1 and 2. Later in this text, we are going to look at a category having manifolds as objects and morphisms. In particular we are going to be interested in the case where the objects are 1-dimensional manifolds and the morphisms 2-dimensional ones. This category will be denoted $\mathbf{2Cob^{or}}$ and the reason these classification theorems are so important to us is because they allows us to completely understand this category.

2.2.1 Classification of 1-dimensional manifolds

For connected and compact 1-dimensional manifolds we have the following result:

Theorem 2.25. [MW97, p. 55] Any connected compact 1-dimensional manifold M is diffeomorphic to either S^1 or the interval $I = [0, 1]$. If M is closed[4] then $M \cong S^1$.

The following lemma will be important for us because in the category $\mathbf{2Cob^{or}}$ this tells us what the objects are (up to isomorphism):

[4] A closed manifold is a compact manifold with empty boundary ($\partial M = \varnothing$).

Lemma 2.26. Any closed 1-dimensional manifold with m connected components is diffeomorphic to the m-fold disjoint union of S^1.

Proof. It follows at once from Theorem 2.25. □

2.2.2 Classification of surfaces

The classification theorem for surfaces requires significantly more work and will only be stated here. The classification theorem of surfaces[5] up to diffeomorphism can be realized by Morse theory. This is done in detail by Hirsch in [Hir97]. An important construction is what is called handle attachment where we glue "handles" onto a manifold to get something new. For example the following illustration shows how a handle is attached to the 2-sphere:

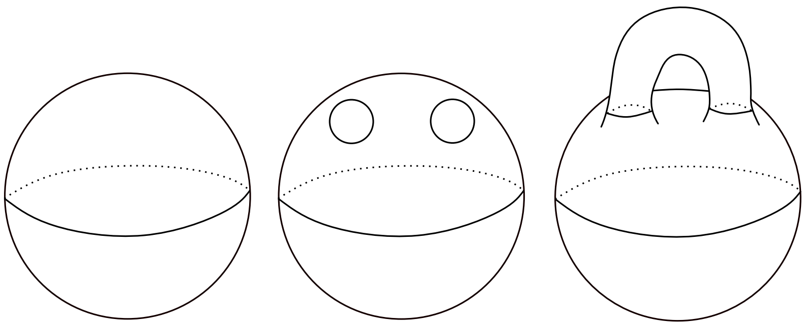


Figure 2.2.1: Gluing a handle to a sphere.

Imagine we repeat this process a finite number of times. We then end up with a manifold with as many handles, or holes, we want.

We start with a manifold M and an embedding of two copies of the closed disk.

$$f : S^0 \times D^2 \hookrightarrow \mathrm{Int}(M)$$

Construct a new manifold $M[f]$ as M with a handle added as follows

$$M[f] = [M - \mathrm{Int}(f(S^0 \times D^2))] \sqcup_{\partial f} [0,1] \times S^1$$

where $\sqcup_{\partial f}$ means that we glue the two ends of the cylinder $[0,1] \times S^1$ along the boundaries of the two copies of D^2 embedded in M (compare to the illustration).[6] It can be shown that this construction gives a smooth manifold and is unique up to diffeomorphism.

Definition 2.27. Let p be an non-negative integer. An orientable surface has *genus* p if it is diffeomorphic to one which can be constructed by attaching handles to S^2 p times. In other words: there exist manifolds $M_0, M_1, \ldots, M_p$ and if $p > 0$, embeddings

$$f_i : S^0 \times D^2 \hookrightarrow \mathrm{Int}(M_{i-1})$$

[5] A *surface* is a 2-manifold which is closed.

[6] By ∂f we mean f restricted to the boundary of its domain.

such that $M_0 \cong S^2$, $M_i \cong M_{i-1}[f_i]$ and $M_p \cong M$ (where $i = 1, \dots, p$).

Theorem 2.28. [Hir97] If M is a connected orientable surface, then M is an orientable surface of genus p for some $p \geq 0$.

2.2.3 Some important consequences

Let M be an oriented and compact 2-manifold with boundary, and let C be a connected component of ∂M. Since C is diffeomorphic to S^1 we have a diffeomorphism $\phi : S^1 \xrightarrow{\cong} C$. Now we can glue the closed unit disk D^2 onto C along the boundary $S^1 = \partial D^2$ via ϕ. If we do this for every component we end up with a closed surface. This gives us the following lemma:

Lemma 2.29. Let M be an oriented and compact 2-manifold with boundary, and let m be the number of connected components in ∂M. Then, M is an orientable surface of genus $p \geq 0$ with m open disks removed.

Since the boundary of a manifold is the union of its incoming boundaries and its outgoing boundaries, we get the following important corollary:

Corollary 2.30. Let M be an oriented and compact 2-manifold with boundary. Then M is determined (up to diffeomorphism) by the genus, the number of in-boundary components and the number of out-boundary components.

The reason why this corollary will become important for us is because it will enable us to completely understand the morphisms in the category $\mathbf{2Cob^{or}}$.

Remark 2.31. We regard the empty set as a n-dimensional manifold and denote this manifold by $\emptyset_n$. The empty disjoint union is the empty manifold.

3 Algebra

3.1 Preliminaries

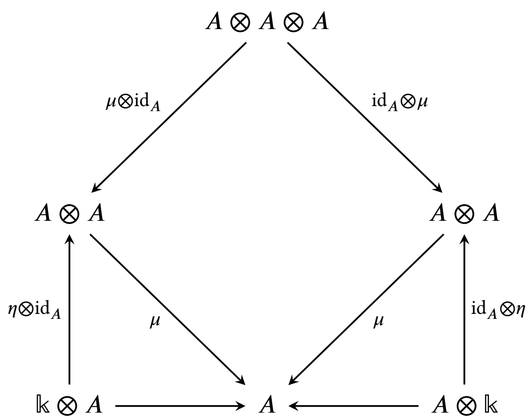
3.1.1 $\Bbbk$-Algebras

We now recall the definition of an algebra over a field $\Bbbk$. For our needs, we only care about associative algebras.

Definition 3.1. A $\Bbbk$-*algebra* is a $\Bbbk$-vector space A endowed with two $\Bbbk$-linear maps

$$\mu\colon\ A \otimes A \to A \quad \text{and} \quad \eta\colon\ \Bbbk \to A,$$

called multiplication and unit, respectively, such that the diamond and the two triangles in the following diagram commute



Here the bottom maps are the canonical isomorphisms defined by scalar multiplication. That is,

$$\lambda \otimes a \mapsto \lambda a \mapsfrom a \otimes \lambda.$$

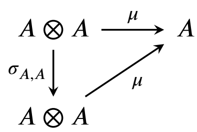
We will denote $1_A := \eta(1_\Bbbk)$.

Remark 3.2. Note that left and right distributivity is captured by the tensor products. What we mean by this is that when we define μ as a linear map from the tensor product (instead of the direct product), we automatically have that: for elements $a, b, c \in A$,

$$\mu(a \otimes (b+c)) \overset{(1)}{=} \mu(a \otimes b + a \otimes c) \overset{(2)}{=} \mu(a \otimes b) + \mu(a \otimes c).$$

At (1) we used a basic property of the tensor product, and at (2) we used that μ is linear. We should also note that we sometimes use the shorthand notation ab when we mean $\mu(a \otimes b)$. Using this notation the relation above becomes $a(b+c) = ab + ac$ which is the familiar left-distributivity. Right-distributivity is captured in a similar way.

Definition 3.3. Let A be an algebra and let $\sigma_{A,A} : A \otimes A \to A \otimes A$ be the map defined by $a \otimes b \mapsto b \otimes a$. Then we say that A is *commutative* if the following diagram commute



On elements that is: $\mu(a \otimes b) = \mu(b \otimes a)$. Using shorthand notation for multiplication that is simply $ab = ba$.

Examples 3.4.

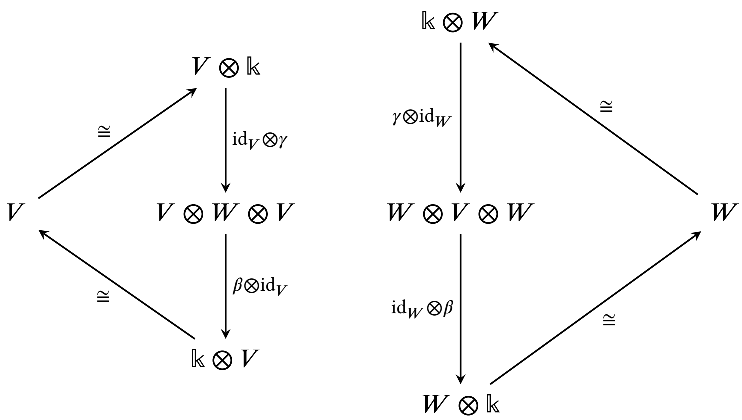
1. The vector space $M_n(\Bbbk)$ of $n \times n$-matrices over $\Bbbk$ is also a ring under matrix multiplication. The compability of these structures make $M_n(\Bbbk)$ into a $\Bbbk$-algebra.
2. The complex numbers $\mathbb{C}$ are a commutative $\mathbb{C}$-algebra with the usual multiplication of complex numbers. This is a special case of the previous example. We could also have considered $\mathbb{C}$ as a vector space over $\mathbb{R}$ to get a $\mathbb{R}$-algebra.
3. Any field $\Bbbk$ is a $\Bbbk$-algebra when given the multiplication in $\Bbbk$.

3.1.2 Non-degenerate pairings

Definition 3.5. Given two vector spaces V and W over $\Bbbk$ we define a *pairing* to be a $\Bbbk$-linear map

$$\beta : V \otimes W \to \Bbbk$$

Definition 3.6. A pairing $\beta : V \otimes W \to \Bbbk$ is called *non-degenerate in V (in W)* if there exists a $\Bbbk$-linear map $\gamma : \Bbbk \to W \otimes V$, called a *copairing*, such that the composition in the left diagram (right diagram) is the identity on V (on W).



Lemma 3.7. [Koc04, p. 82] If the pairing $\beta : V \otimes W \to \Bbbk$ is non-degenerate in both V and W then the copairings, which makes the above diagrams commute, are identical.

Construction. Given a pairing $\beta : V \otimes W \to \Bbbk$ we get two canonically induced maps to the duals, namely:

$$\begin{aligned} \beta_l : W &\to V^* \\ w &\mapsto [\beta(-, w) : V \to \Bbbk] \end{aligned} \qquad \text{and} \qquad \begin{aligned} \beta_r : V &\to W^* \\ v &\mapsto [\beta(v, -) : W \to \Bbbk] \end{aligned}$$

Lemma 3.8. [Koc04, p. 83] Let $\beta : V \otimes W \to \Bbbk$ be a pairing. Then the following are equivalent:

1. β is non-degenerate in W.
2. W is finite-dimensional and the induced map $\beta_l : W \to V^*$ is injective.

(Similarly, β is non-degenerate in V if and only if V is of finite dimension and β_r is injective.)

By the above lemma, we immediately have the following lemma and its corollary:

Lemma 3.9. [Koc04, p. 85] Let $\beta : V \otimes W \to \Bbbk$ be a pairing. Then the following statements are equivalent:

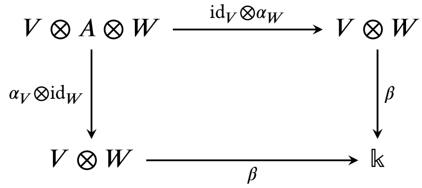
1. β is non-degenerate.
2. $\beta_l : W \to V^*$ is an isomorphism.
3. $\beta_r : V \to W^*$ is an isomorphism.

Corollary 3.10. Let V and W be finite dimensional vector spaces of equal dimension. Then the following are equivalent:

1. β is non-degenerate.
2. $\forall v \in V, \beta(v, w) = 0 \implies w = 0$.
3. $\forall w \in W, \beta(v, w) = 0 \implies v = 0$.

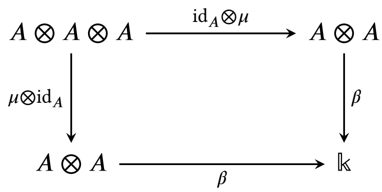
Proof. Observe that (2) and (3) are the same as saying that the induced maps are injective. Since V and W are of equal dimension this implies that β_l and β_r are isomorphisms. $\square$

Definition 3.11. Let A be a $\Bbbk$-algebra and let V and W be left and right A-modules, respectively. A pairing $\beta : V \otimes W \to \Bbbk$ is said to be *associative* if the following diagram commutes:



Here α_V and α_W are the actions of A on V and W.

Remark 3.12. Now, every algebra A can be viewed as an A-module by letting the action be the multiplication in A. In this case, a pairing $\beta:\ A\otimes A \to \Bbbk$ being associative corresponds to having the following commutative diagram:



On elements this translates into the equation $\beta(xy, z) = \beta(x, yz)$ for all $x, y, z \in A$.

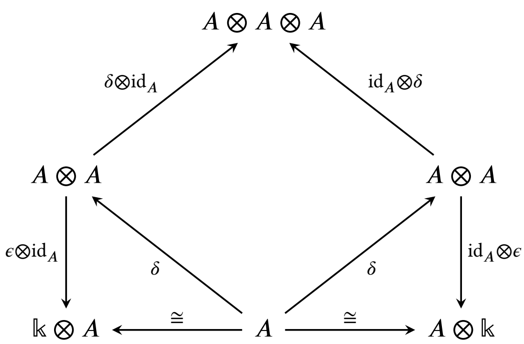
3.1.3 Coalgebras

The definition of a coalgebra is similar to the one for an algebra.

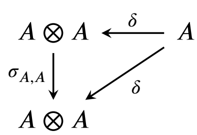
Definition 3.13. A *coalgebra* over $\Bbbk$ is a $\Bbbk$-vector space A endowed with two $\Bbbk$-linear maps

$$\delta:\ A \to A \otimes A \quad \text{and} \quad \epsilon:\ A \to \Bbbk$$

called comultiplication and counit respectively, such that the diamond and the two triangles in the following diagram commutes.



Definition 3.14. Let A be a coalgebra and let $\sigma_{A,A}:\ A \otimes A \to A \otimes A$ be the map defined by $a \otimes b \mapsto b \otimes a$. Then we say that A is *cocommutative* if the following diagram commute



3.2 Frobenius algebras

3.2.1 Definition and examples

Definition 3.15. A *Frobenius algebra* is a finite dimensional $\Bbbk$-algebra A endowed with a $\Bbbk$-linear functional $\epsilon : A \to \Bbbk$ where $\ker(\epsilon) \subset A$ contains no nontrivial left[7] ideals. We call ϵ a *Frobenius form.* If $\epsilon(ab) = \epsilon(ba)$ for all $a, b \in A$, we say that ϵ is *central* and that A is *symmetric.*

We now give another definition of a Frobenius algebra. It is not obvious that this is equivalent, but we will prove it later in this section.

Definition 3.16. A *Frobenius algebra* is a finite dimensional $\Bbbk$-algebra A endowed with an associative non-degenerate pairing $\beta : A \otimes A \to \Bbbk$ called the *Frobenius pairing*. If $\beta(a, b) = \beta(b, a)$ for all $a, b \in A$, we say that A is *symmetric.*

Lemma 3.17. The matrix algebra $M_n(\Bbbk)$ with the Frobenius form given by the trace of a matrix is a Frobenius algebra.

$$\begin{aligned} \epsilon : M_n(\Bbbk) &\to \Bbbk \\ A &\mapsto \mathrm{tr}(A) \end{aligned}$$

We give a short proof of why this is indeed a Frobenius form.

Proof. Suppose that $I \subsetneq \ker(\epsilon)$ is an ideal and $A = (a_{ij}) \in I$. Since I is an ideal we have that $E_{ji}A \in I$ as well.[8] Since $I \subsetneq \ker(\epsilon)$, we have that $0 = \mathrm{tr}(E_{ji}A) = a_{ij}$. This is true for all $1 \leq i, j \leq n$ so $A = 0$ and therefore $I = (0)$. Hence, $\ker(\epsilon)$ only contains trivial ideals and ϵ is a Frobenius form. □

Note that the trace enjoys the property that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. This is to say that $M_n(\Bbbk)$ is a symmetric Frobenius algebra that is not commutative since $AB \neq BA$ in general.

Since the result we seek is a connection between topology and algebra, it will be convenient to introduce a graphical notation. Advantages of using this notation includes:

1. It allows us to see the topological nature of Frobenius algebras more clearly, which will be justified later by the main theorem.

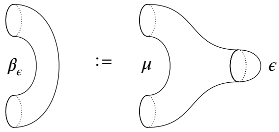
2. It can make proofs cleaner than when using commutative diagrams.

3. It is fun!

Given a Frobenius form $\epsilon : A \to \Bbbk$ we can construct a pairing $\beta_\epsilon : A \otimes A \to \Bbbk$ by defining $\beta := \epsilon \circ \mu$ where μ is the multiplication in A. Using graphical notation we picture this definition as follows:

[7] Here we could just as well required $\ker(\epsilon)$ to contain no nontrivial *right* ideal. A proof of this is given in [Koc04, p. 95].

[8] By E_{ij} we mean the matrix with 1 in the i, j-spot and 0 everywhere else.



We will see that this pairing induced from ϵ is non-degenerate. In fact this is true if and only if ϵ is a Frobenius form:

Lemma 3.18. Let $\epsilon : A \to \Bbbk$ and $\beta := \epsilon \circ \mu : A \otimes A \to \Bbbk$. Then $\ker(\epsilon)$ contains no non-trivial left ideals if and only if β is non-degenerate.

Proof. Suppose β is non-degenerate and $I \subset \ker(\epsilon)$ is a left ideal. Let $a \in I$. We then have that for every $x \in A$

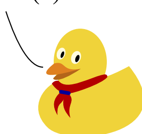
$$0 = \epsilon(xa) = \beta(x, a)$$

since $xa \in I$. By non-degeneracy of β we have that $a = 0$ and hence $I = (0)$. Conversely, suppose β is degenerate, so there exists some $y \neq 0$ in A such that for every $x \in A$

$$0 = \beta(x, y) = \epsilon(xy).$$

So Ay is a left ideal contained in the kernel of ϵ which is trivial since $y = 1_A y \in Ay$.

But that's a contraducktion since $\ker(\epsilon)$ contains no non-trivial left ideals!

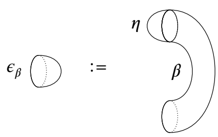


□

Conversely, if we are given a non-degenerate pairing $\beta : A \otimes A \to \Bbbk$ we can define the form ϵ_β to be:

$$\begin{aligned} \epsilon_\beta : A &\to \Bbbk \\ x &\mapsto \beta(x, 1_A), \end{aligned}$$

or in other words $\epsilon_\beta := \beta \circ (\mathrm{id}_A \otimes \eta)$ where η is the unit in A. Using graphical notation we can picture it as follows:



Lemma 3.19. Let $\beta : A \otimes A \to \Bbbk$ be a pairing and define the form $\epsilon_\beta := \beta \circ (\mathrm{id}_A \otimes \eta)$. Then we have that β is non-degenerate if and only if $\ker(\epsilon_\beta)$ contains no non-trivial left ideals.

Proof. Suppose that I is a left ideal contained in $\ker(\epsilon_\beta)$ with $I \neq A$. Let $a \in I$. Then we have that for every $x \in A$, $xa \in I$. By associativity of β we have that

$$0 = \epsilon_\beta(xa) = \beta(xa, 1_A) = \beta(x, a).$$

Since β is non-degenerate, $a = 0$ and hence I is trivial. Conversely, assume that $\ker(\epsilon_\beta)$ contains no non-trivial left ideals. Suppose that β is degenerate. That is, there exist some non-zero element $y \in A$ so that for every $x \in A$ we have that

$$\beta(x, y) = 0.$$

Thus, we have that

$$0 = \beta(x, y) = \beta(xy, 1_A) = \epsilon_\beta(xy).$$

This is to say that the left ideal $Ay \neq (0)$ is contained in $\ker(\epsilon_\beta)$ which contradicts the assumption that $\ker(\epsilon_\beta)$ contains no non-trivial left ideals. Hence, β is non-degenerate. □

We now prove the equivalence of the two definitions:

Lemma 3.20. Definition 3.15 and definition 3.16 are equivalent.

Proof. Lemma 3.18 and 3.19 gave us that every Frobenius form induces a Frobenius pairing and vice versa. We now complete the proof by observing that these constructions gives us an one-to-one correspondence between Frobenius forms and Frobenius pairings:

$$\begin{aligned} \{\text{Frobenius forms}\} &\overset{\text{1-1}}{\longleftrightarrow} \{\text{Frobenius pairings}\} \\ \epsilon &\longmapsto [x \otimes y \mapsto \epsilon(xy)] \\ [x \mapsto \beta(x, 1_A)] &\longleftarrow\!\shortmid\ \beta \end{aligned}$$

Observe that these maps are inverses to one another:

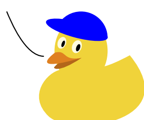
$$\epsilon \mapsto [x \otimes y \mapsto \epsilon(xy)] \mapsto [x \mapsto \epsilon(x1_A) = \epsilon(x)] = \epsilon$$

$$\beta \mapsto [x \mapsto \beta(x, 1_A)] \mapsto [x \otimes y \mapsto \beta(xy, 1_A) = \beta(x, y)] = \beta$$

Note that we used the associativity of β to give us the equality $\beta(xy, 1_A) = \beta(x, y)$. □

3.2.2 Group algebra

Here's a fun way to construct a Frobenius algebra!



Start with a finite group $(G, \cdot) = \{g_1, g_2, \dots, g_n\}$, where g_1 is the identity element in G. Now let $\Bbbk[G]$ consist of all $\Bbbk$-linear combinations of elements in G. That is, the elements of $\Bbbk[G]$ look like this:

$$a = \sum_{i=1}^{n} \alpha_i g_i, \quad \alpha_i \in \Bbbk.$$

The multiplication is given by the group operation of G as follows: We can define it on the basis elements as

$$\begin{aligned} \Bbbk[G] \otimes \Bbbk[G] &\to \Bbbk[G] \\ g_i \otimes g_j &\mapsto g_i \cdot g_j \end{aligned}$$

And here is a $\Bbbk$-linear functional on $\Bbbk[G]$:

$$\begin{aligned} \epsilon : \Bbbk[G] &\to \Bbbk \\ g_1 &\mapsto 1 \\ g_j &\mapsto 0, \qquad \text{for } j \neq 1. \end{aligned}$$

Let us prove that this form is in fact a Frobenius form.

Proof. Using that ϵ is $\Bbbk$-linear by construction we have that

$$\ker(\epsilon) = \{\alpha_2 g_2 + \dots + \alpha_n g_n | \alpha_i \in \Bbbk\}.$$

Let $I \subsetneq \ker(\epsilon)$ be an ideal and suppose $a \in I$. Then

$$a = \alpha_2 g_2 + \dots + \alpha_n g_n.$$

Multiplying by g_j^{-1} on both sides gives

$$g_j^{-1} \cdot a = \alpha_2 \underbrace{g_j^{-1} \cdot g_2}_{\neq g_1} + \dots + \alpha_j \underbrace{g_j^{-1} \cdot g_j}_{= g_1} + \dots + \alpha_n \underbrace{g_j^{-1} \cdot g_n}_{\neq g_1}$$

which is also in I (since I is a left ideal). Observe that for all $j = 2, 3 \dots, n$ we have

$$\begin{aligned} 0 &= \epsilon(\alpha_2 g_j^{-1} \cdot g_2 + \dots + \alpha_j g_j^{-1} \cdot g_j + \dots + \alpha_n g_j^{-1} \cdot g_n) \\ &= \underbrace{\epsilon(\alpha_2 g_j^{-1} \cdot g_2)}_{=0} + \dots + \underbrace{\epsilon(\alpha_j g_j^{-1} \cdot g_j)}_{=\alpha_j} + \dots + \underbrace{\epsilon(\alpha_n g_j^{-1} \cdot g_n)}_{=0} = \alpha_j, \end{aligned}$$

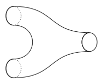
and hence $a = 0$ and $I = (0)$. □

Example 3.21. Take G to be the multiplicative group of integers modulo a prime p.

$$G = \left(\mathbb{Z}/p\mathbb{Z}\right)^{\times} = \{1, 2, \dots, p-1\}$$

and let $A = \Bbbk[G]$ be the group algebra of G over $\Bbbk$. Then A is a Frobenius algebra over $\Bbbk$.

Let us explicitly calculate the case when $p = 3$. Two elements can be written as $a = \alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2}$ and $b = \beta_1 \cdot \mathbf{1} + \beta_2 \cdot \mathbf{2}$ where $\alpha_i, \beta_i \in \Bbbk$. We now list the algebraic expressions together with their graphical representation on the left:



$$\mu\colon\ A \otimes A \to A$$
$$\alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2} \otimes \beta_1 \cdot \mathbf{1} + \beta_2 \cdot \mathbf{2} \mapsto (\alpha_1\beta_1 + \alpha_2\beta_2) \cdot \mathbf{1} + (\alpha_1\beta_2 + \alpha_2\beta_1) \cdot \mathbf{2}$$

$$\epsilon\colon\ A \to \Bbbk$$
$$\alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2} \mapsto \alpha_1$$

$$\eta\colon\ \Bbbk \to A$$
$$1_\Bbbk \mapsto 1_\Bbbk \cdot \mathbf{1} =\colon 1_A$$

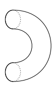
This is how we obtain the explicit expression for μ: by the properties of the tensor product we have that

$$\alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2} \otimes \beta_1 \cdot \mathbf{1} + \beta_2 \cdot \mathbf{2} = \alpha_1\beta_1 \mathbf{1} \otimes \mathbf{1} + \alpha_1\beta_2 \mathbf{1} \otimes \mathbf{2} + \alpha_2\beta_1 \mathbf{2} \otimes \mathbf{1} + \alpha_2\beta_2 \mathbf{2} \otimes \mathbf{2}.$$

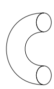
Using the linearity of μ we then obtain that

$$\begin{aligned}\mu(a \otimes b) &= \mu(\alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2} \otimes \beta_1 \cdot \mathbf{1} + \beta_2 \cdot \mathbf{2}) \\ &= \alpha_1\beta_1\mu(\mathbf{1} \otimes \mathbf{1}) + \alpha_1\beta_2\mu(\mathbf{1} \otimes \mathbf{2}) + \alpha_2\beta_1\mu(\mathbf{2} \otimes \mathbf{1}) + \alpha_2\beta_2\mu(\mathbf{2} \otimes \mathbf{2}) \\ &= \alpha_1\beta_1\mathbf{1} + \alpha_1\beta_2\mathbf{2} + \alpha_2\beta_1\mathbf{2} + \alpha_2\beta_2\mathbf{1} \\ &= (\alpha_1\beta_1 + \alpha_2\beta_2) \cdot \mathbf{1} + (\alpha_1\beta_2 + \alpha_2\beta_1) \cdot \mathbf{2}.\end{aligned}$$

Next, we construct the non-degenerate pairing $\beta := \epsilon \circ \mu$ and its corresponding co-pairing γ.



$$\beta\colon\ A \otimes A \to \Bbbk$$
$$\alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2} \otimes \beta_1 \cdot \mathbf{1} + \beta_2 \cdot \mathbf{2} \mapsto \alpha_1\beta_1 + \alpha_2\beta_2$$



$$\gamma\colon\ \Bbbk \to A \otimes A$$
$$1_\Bbbk \mapsto \mathbf{1} \otimes \mathbf{1} + \mathbf{2} \otimes \mathbf{2}$$

We check that this is in fact non-degenerate by checking that the required compositions are the identity on A:

$$A \xrightarrow{\cong} \Bbbk \otimes A \xrightarrow{\gamma \otimes \mathrm{id}_A} A \otimes A \otimes A \xrightarrow{\mathrm{id}_A \otimes \beta} A \otimes \Bbbk \xrightarrow{\cong} A$$

$$a \mapsto 1_\Bbbk \otimes a \mapsto (\mathbf{1}\otimes\mathbf{1}+\mathbf{2}\otimes\mathbf{2})\otimes a = \mathbf{1}\otimes\mathbf{1}\otimes a+\mathbf{2}\otimes\mathbf{2}\otimes a \mapsto \mathbf{1}\otimes\alpha_1+\mathbf{2}\otimes\alpha_2 \stackrel{(1)}{=} a\otimes 1_\Bbbk \mapsto a.$$

The equality (1) is justified as follows:

$$\begin{aligned}\mathbf{1} \otimes \alpha_1 + \mathbf{2} \otimes \alpha_2 &= \alpha_1 \mathbf{1} \otimes 1_\Bbbk + \alpha_2 \mathbf{2} \otimes 1_\Bbbk \\ &= (\alpha_1 \mathbf{1} + \alpha_2 \mathbf{2}) \otimes 1_\Bbbk \\ &= a \otimes 1_\Bbbk.\end{aligned}$$

The equality (2), in the other composition below, can be justified similarly.

$$A \xrightarrow{\cong} A \otimes \Bbbk \xrightarrow{\mathrm{id}_A \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes \mathrm{id}_A} \Bbbk \otimes A \xrightarrow{\cong} A$$

$$a \mapsto a\otimes 1_\Bbbk \mapsto a\otimes(\mathbf{1}\otimes\mathbf{1}+\mathbf{2}\otimes\mathbf{2}) = a\otimes\mathbf{1}\otimes\mathbf{1}+a\otimes\mathbf{2}\otimes\mathbf{2} \mapsto \alpha_1\otimes\mathbf{1}+\alpha_2\otimes\mathbf{2} \stackrel{(2)}{=} 1_\Bbbk\otimes a \mapsto a.$$

We shall now see where the expression for the copairing γ came from. Let $G = \{g_1, \dots, g_n\}$ be a finite group where g_1 is the identity element. Let A be the group algebra $\Bbbk[G]$ and define $\beta := \epsilon \circ \mu : A \otimes A \to \Bbbk$. Two elements a and b in A can be written as

$$a = \sum_{i=1}^{n} \alpha_i g_i \quad \text{and} \quad b = \sum_{j=1}^{n} \beta_j g_j.$$

Straightforward calculation shows that

$$\beta(a, b) = \sum_{\substack{i,j \\ \text{s.t. } g_i g_j = g_1}} \alpha_i \beta_j.$$

Now, define the copairing γ as follows:

$$\begin{aligned}\gamma : \Bbbk &\to A \otimes A \\ 1_\Bbbk &\mapsto \sum_{k=1}^{n} g_k^{-1} \otimes g_k.\end{aligned}$$

This is exactly the copairing we need for the required compositions to be the identity. Let us calculate one of these compositions. (The other one is similar.)

$$A \xrightarrow{\cong} A \otimes \Bbbk \xrightarrow{\mathrm{id}_A \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes \mathrm{id}_A} \Bbbk \otimes A \xrightarrow{\cong} A$$

$$a \mapsto a \otimes \gamma(1_\Bbbk) = \sum_{k,i=1}^{n} \alpha_i g_i \otimes g_k^{-1} \otimes g_k \mapsto \sum_{k,i=1}^{n} \alpha_i \delta_{i,k} \otimes g_k = \sum_{i=1}^{n} \alpha_i g_k \otimes 1_\Bbbk \mapsto a$$

(Note that we here used properties of the tensor product move sums and scalars around.) Here $\delta_{i,k}$ is the Kronecker delta function defined by

$$\delta_{i,k} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

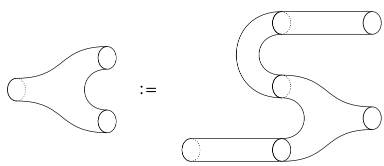
The reason why we get the Kronecker delta here can be justified by the following calculation:

$$\begin{aligned} \beta(g_i \otimes g_k^{-1}) &= \epsilon \circ \mu(g_i \otimes g_k^{-1}) \\ &= \epsilon(g_i g_k^{-1}) \\ &= \begin{cases} 1 & \text{if } g_i g_k^{-1} = g_1 \iff g_i = g_k \iff i = k \\ 0 & \text{otherwise} \end{cases} \\ &= \delta_{i,k}. \end{aligned}$$

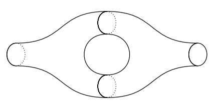
We shall later see that any Frobenius algebra can be endowed with a comultiplicative structure. That is, a comultiplication map and a counit. The following construction, using the multiplication and the copairing, is also how we will construct the comultiplicative structure in general for any Frobenius algebra:

$$\begin{aligned} \delta := (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma) : \; & A \to A \otimes A \\ & a \mapsto (\alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2}) \otimes \mathbf{1} + (\alpha_2 \cdot \mathbf{1} + \alpha_1 \cdot \mathbf{2}) \otimes \mathbf{2} \end{aligned}$$

Graphically we can think of this composition in the following way:



We define the *handle operator* to be the map $H = \mu \circ \delta : A \to A$. The reason behind the name becomes clear when we look at the graphical representation:

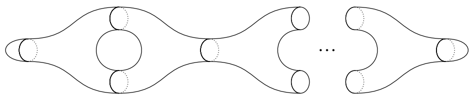


Since we know both μ and γ, calculation gives that $H(a) = 2a$. In other words, for $p = 3$, the handle operator is multiplication by 2. Let us write it out:

$$\begin{aligned} \mu \circ \delta(a) &= \mu \circ \delta(\alpha_1 \mathbf{1} + \alpha_2 \mathbf{2}) \\ &= \mu((\alpha_1 \cdot \mathbf{1} + \alpha_2 \cdot \mathbf{2}) \otimes \mathbf{1} + (\alpha_2 \cdot \mathbf{1} + \alpha_1 \cdot \mathbf{2}) \otimes \mathbf{2}) \\ &= \alpha_1 \mu(\mathbf{1} \otimes \mathbf{1}) + \alpha_2 \mu(\mathbf{2} \otimes \mathbf{1}) + \alpha_2 \mu(\mathbf{1} \otimes \mathbf{2}) + \alpha_1 \mu(\mathbf{2} \otimes \mathbf{2}) \\ &= \alpha_1 \mathbf{1} + \alpha_2 \mathbf{2} + \alpha_2 \mathbf{2} + \alpha_1 \mathbf{1} \end{aligned}$$

$$= 2(\alpha_1 \mathbf{1} + \alpha_2 \mathbf{2}) = 2a$$

If we compose n handle operators with the unit and the counit we get a picture of an n-holed torus:



This corresponds to the map $\epsilon \circ H \circ \ldots \circ H \circ \eta : \Bbbk \to \Bbbk$. Again we know all the involved maps, so we can calculate the image of $1_\Bbbk$ under this map:

$$1_\Bbbk \overset{\eta}{\mapsto} 1_A \overset{H}{\mapsto} 2 \cdot 1_A \overset{H}{\mapsto} 2^2 \cdot 1_A \overset{H}{\mapsto} \ldots \overset{H}{\mapsto} 2^n \cdot 1_A \overset{\epsilon}{\mapsto} 2^n. \tag{1}$$

In other words, this map counts the number of holes. We will get back to this example when we get to TQFTs.

Remark 3.22. In general, taking $G = \mathbb{Z}/_{p\mathbb{Z}}$ for any prime p, we have the following comultiplication on the group algebra $A = \Bbbk[G]$:

$$\begin{aligned} \delta : A &\to A \otimes A \\ 1_A &\mapsto \sum_{m=1}^{p-1} \mathbf{m}^{-1} \otimes \mathbf{m}. \end{aligned}$$

Straightforward calculation similar to the one for $p = 3$ gives that composition of g handles $\epsilon \circ H \circ \ldots \circ H \circ \eta : \Bbbk \to \Bbbk$ is the map $1_\Bbbk \mapsto (p-1)^g$.

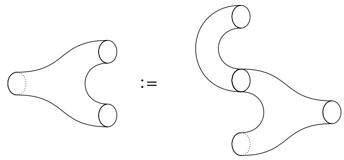
Remark 3.23. A Frobenius algebra A which have the property that the handle operator is the identity is called *special.*

3.2.3 A natural coalgebra structure on Frobenius algebras and the Frobenius relation

We saw in the example with the group algebra that given multiplication, unit and taking the Frobenius form as the counit, we where able to construct a non-degenerate pairing, giving us a copairing which we then used to construct the comultiplication. In fact, given any Frobenius algebra (A, ϵ) it can be given an unique coalgebra structure where the Frobenius form $\epsilon : A \to \Bbbk$ acts as the counit in exactly the way we did it in the group algebra example.

There is another way to characterize Frobenius algebras and that is by what is called the Frobenius relation (also called the Frobenius law). It allows us to generalize the concept of Frobenius algebras to Frobenius objects in general monoidal categories.

Let A be a Frobenius algebra with multiplication μ, unit η and Frobenius form ϵ. Define the comultiplication $\delta := (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \gamma)$, where γ is the copairing corresponding to the non-degenerate pairing $\beta := \epsilon \circ \mu$. We can represent this graphically as follows:

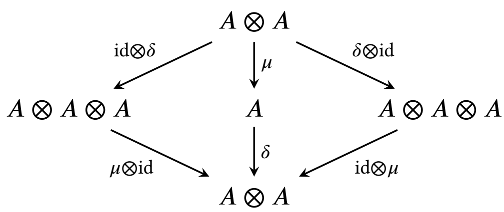


(Notice how we allow ourselves to omit the identity cylinders for simplicity. But for this to make sense as say a diagram we would need to put them in again.)

We will soon prove that when defining the comultiplication like this, then the following relation is satisfied:

$$(\mu \otimes \mathrm{id})\circ(\mathrm{id} \otimes \delta) = \delta\circ\mu = (\mathrm{id} \otimes \mu)\circ(\delta \otimes \mathrm{id}).$$

As a commutative diagram that is:



Graphically we can think of it like this:

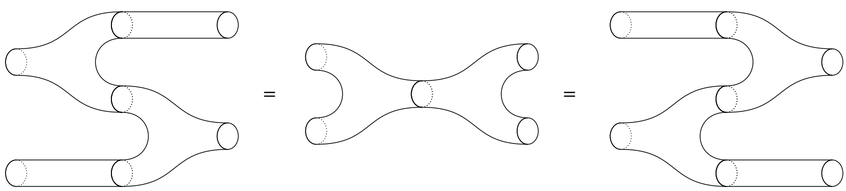


Figure 3.2.1: Graphical representation of the Frobenius relation.

Definition 3.24. We call the relation

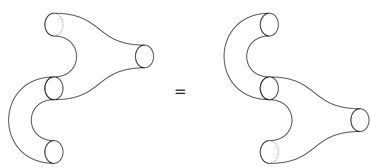
$$(\mu \otimes \mathrm{id})\circ(\mathrm{id} \otimes \delta) = \delta\circ\mu = (\mathrm{id} \otimes \mu)\circ(\delta \otimes \mathrm{id}),$$

the *Frobenius relation.*

Lemma 3.25. [Koc04, p. 116]

$$(\mu \otimes \mathrm{id}_A)\circ(\mathrm{id}_A \otimes \gamma) = (\mathrm{id}_A \otimes \mu)\circ(\gamma \otimes \mathrm{id}_A)$$

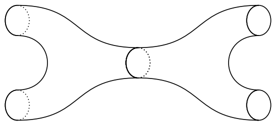
Graphically that is:



Lemma 3.26. With the comultiplication δ defined as above, δ and μ satisfy the Frobenius relation.

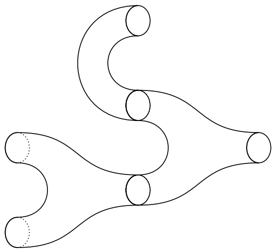
Proof. We will do this proof graphically and diagrammatically side by side to show the simplicity of using the graphical notation. At the same time it demonstrates the topological nature of Frobenius algebras which is what this text is really all about. On the left we can observe that the pictures we get from the graphical notation are diffeomorphic if we interpret them as manifolds with boundary. And on the right we have the commutative diagrams corresponding to the graphical notation on the left. This is a sneak peek at one of the consequences of our main theorem.

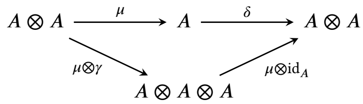
(1)



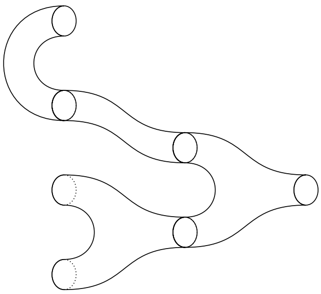
$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\delta} A \otimes A$$

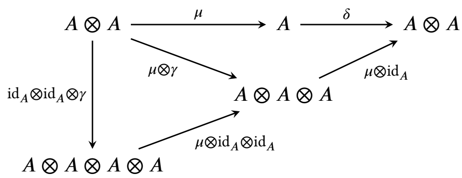
(2)



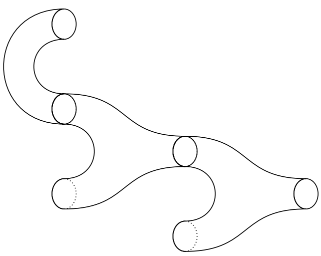


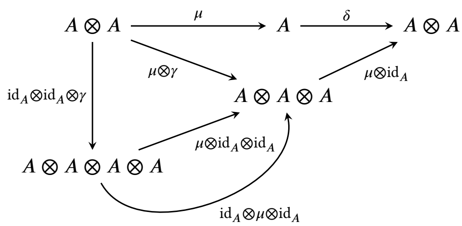
(3)



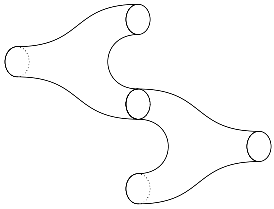


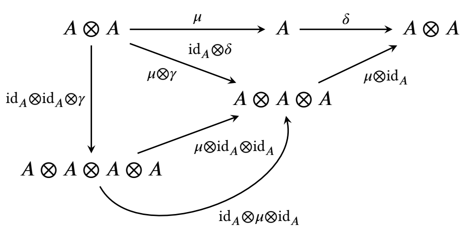
(4)





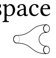
(5)



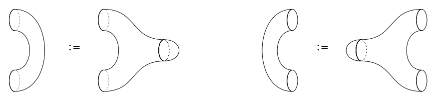


In the first step we inserted the definition of δ. From (2) to (3) we inserted the identity so that it gets easier to see how we get the arrow in (4) from the associativity of μ. In the last step are just using the definition of δ again. To get the mirrored Frobenius relation we just use lemma 3.25 to express δ. □

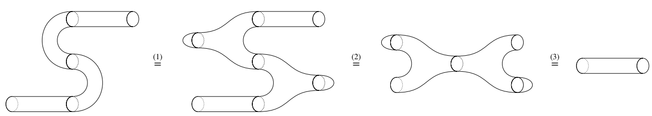
Lemma 3.27. [Koc04, p. 118] Let A be a Frobenius algebra with Frobenius form ϵ. Then there exists a unique comultiplication δ whose counit is ϵ and which satisfies the Frobenius relation and δ is coassociative.

Lemma 3.28. [Koc04, p. 119] Let A be a vector space equipped with multiplication $\mu =$ , unit $\eta =$ , comultiplication $\delta =$  and counit $\epsilon =$ . If the Frobenius relation holds, then A is a Frobenius algebra.

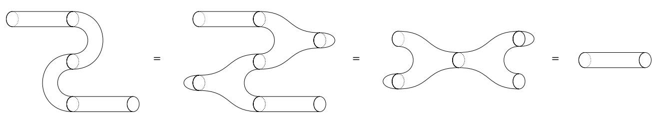
Proof. Let us to this proof entirely with graphical notation. Begin by defining the pairing β and the corresponding copairing γ as follows:



We now verify definition 3.16 graphically by showing that β is non-degenerate:



For (1) we inserted the definition of β and γ. For (2) we use the Frobenius relation and finally for (3) we use the unit and counit relation. Equivalently we get non-degeneracy in the other factor by using the mirrored Frobenius relation:



Hence β, as we defined it, is non-degenerate and by lemma 3.8, A is finite dimensional. Next we will show that μ is associative and δ is coassociative. First we observe the following relation:

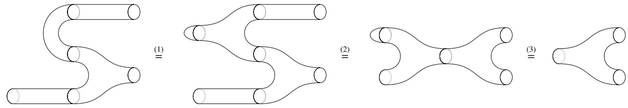


Figure 3.2.2: Relation (a)

For (1) we insterted the definition of the copairing γ. For (2) we used the Frobenius relation and for (3) we used the unit relation. Similarly we can express μ in terms of δ and β:

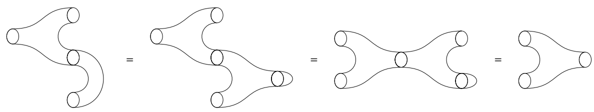
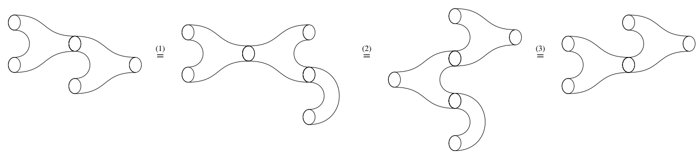
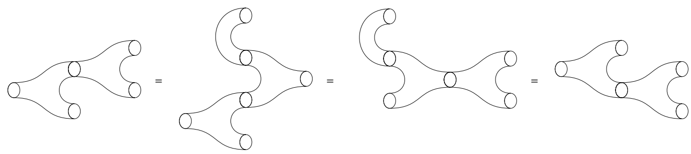


Figure 3.2.3: Relation (b)

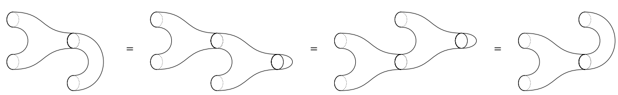
Notice how we now have gone rogue and omitted identity cylinders. We will continue doing this for the rest of this proof because of laziness. Using these relation it is easy to show that μ is associative:



For (1) we used relation (b) above. For (2) we used the Frobenius relation and for (3) we used relation (b) again. Similarly for the coassociativity of δ using relation (a) and the Frobenius relation:



The fact that β is associative follows at once from the associativity of μ:



We conclude that A is indeed a Frobenius algebra with the counit as its Frobenius form. $\square$

These graphical proofs gives a strong indication of the topological nature of Frobenius algebras. That is, all the relations we have seen holds topologically as well. And we shall see that this is the key observation for proving the main result.

3.2.4 The category of Frobenius algebras

Frobenius algebras form a category (a subcategory of $\mathbf{Vect}_{\Bbbk}$) and we will be particularly interested in the category of commutative Frobenius algebras over a field $\Bbbk$ which we will denote by $\mathbf{cFA}_{\Bbbk}$. A map $f : A \to B$ is a morphism of Frobenius algebras if it is both an algebra morphism and a coalgebra morphism. In other words it is compatible with all the structure, which amounts to requiring that all of the following diagrams commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ {\scriptstyle \mu_A}\big\downarrow & & \big\downarrow{\scriptstyle \mu_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ {\scriptstyle \delta_A}\big\uparrow & & \big\uparrow{\scriptstyle \delta_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \eta_A}\big\uparrow & & \big\uparrow{\scriptstyle \eta_B} \\ \Bbbk & \xrightarrow{\mathrm{id}_{\Bbbk}} & \Bbbk \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \epsilon_A}\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_B} \\ \Bbbk & \xrightarrow{\mathrm{id}_{\Bbbk}} & \Bbbk \end{array}$$

The two diagrams on the left is the same as saying that f is a morphism of algebras. Similarly, the two diagrams on the right amounts to f being a morphism of coalgebras.

Definition 3.29. We define *the category of commutative Frobenius algebras* over $\Bbbk$, denoted $\mathbf{cFA}_{\Bbbk}$, to be the category consisting of commutative Frobenius algebras as its objects and morphisms of Frobenius algebras as its morphisms.

By requiring this much compability with the structure, it turns out that every morphism of Frobenius algebras is in fact an isomorphism:

Lemma 3.30. [Koc04, p. 132] Every morphism of Frobenius algebras is an isomorphism.

The proof is based on the following lemma because any Frobenius algebra is finite dimensional:

Lemma 3.31. If $f : A \to B$ be a morphism of Frobenius algebras, then f is injective.

Proof. Since f is a morphism of Frobenius algebras, it is compatible with the forms. Hence, we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \epsilon_A}\big\downarrow & \swarrow{\scriptstyle \epsilon_B} & \\ \Bbbk & & \end{array}$$

If $a \in \ker(f)$, then $\epsilon_A(a) = \epsilon_B(f(a)) = \epsilon_B(0) = 0$. Thus, $\ker(f)$ is contained in $\ker(\epsilon_A)$ so that $\ker(f) = (0)$ proving the injectivity of f. □

4 Symmetric monoidal categories

4.1 Some elementary definitions

We shall first give some basic definitions which will be important for this section.

Definition 4.1. A functor $F : \mathscr{C} \to \mathscr{D}$ is called *full* if it is surjective on the sets of morphisms. That is, for all pairs of objects $X, Y \in \mathscr{C}$ the map

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{F} \mathrm{Hom}_{\mathscr{D}}(FX, FY)$$

is surjective. When it is injective we say that F is *faithful*. If F is both full and faithful we say that F is *fully faithful*.

Definition 4.2. A functor $F : \mathscr{C} \to \mathscr{D}$ is *essentially surjective* (or *dense*) if for every object $D \in \mathscr{D}$ there exists an object $C \in \mathscr{C}$ and an isomorphism $FC \cong D$.

Definition 4.3. Let $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation α from F to G*, denoted $\alpha : F \Rightarrow G$, is a family of morphisms $\{\alpha_X : FX \to GX\}_{X \in \mathscr{C}}$ such that for every morphism $f : X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\alpha_Y} & GY \end{array}$$

We call α_X the *component of α at X*. If all components of α are isomorphism we call α a *natural isomorphism*.

4.2 Monoidal categories

Roughly speaking, a monoidal category is a category together with a "monoidal product", which we will call the tensor product, and a neutral object. The usual tensor product of vector spaces is a monoidal product in $(\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk)$ where $\Bbbk$ serves as a neutral object since $V \otimes \Bbbk \cong V$. However, we could also turn $\mathbf{Vect}_{\Bbbk}$ into a monoidal category by endowing it with the direct sum $\oplus$ as its product, and the zero dimensional vector space as a neutral object. To avoid confusion, note that when talking about monoidal categories in general we will always denote the monoidal product by $\otimes$. We list some additional examples of monoidal categories before making the concept precise:

Example 4.4.

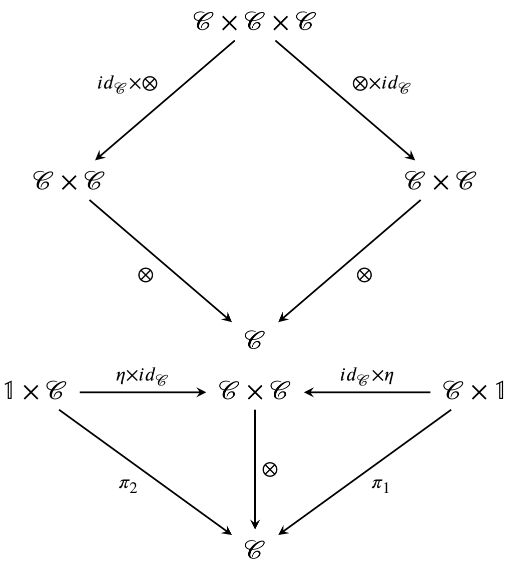
1. The category of sets together with Cartesian product and a set with only one element as a neutral object.

2. The category of topological spaces together with disjoint union and the empty set as the neutral object.

3. The category of R-modules over a commutative ring R together with the usual tensor product $\otimes_R$ of R-modules and R as a neutral object.

Remark 4.5. We will define what is known as a *strict* monoidal category even though these are rare to find in nature. The notion of a monoidal category is weaker and captures actually interesting categories such as the ones in Example 4.4. The price to pay for this weakening is that the definition becomes somewhat more complicated. But luckily for us, we can pretend every monoidal category is a strict one as a consequence of a theorem of MacLane which states that every monoidal category is monoidally equivalent to a strict monoidal category. A more detailed explaination of this is given in [Koc04, p. 154].

Definition 4.6. A *strict monoidal category* $(\mathscr{C}, \otimes, \eta)$ is a category $\mathscr{C}$ endowed with a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a functor $\eta : \mathbb{1} \to \mathscr{C}$ such that the diagrams



commute.[9]

Remark 4.7. Let I denote the image of the single object in $\mathbb{1}$ under η and let $A, B, C \in \mathscr{C}$. Then on the level of objects, the first diagram amounts to saying $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ and the second diagram to $I \otimes A = A = A \otimes I$. Similarly for arrows f, g, h: $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ and $id_I \otimes f = f = f \otimes id_I$. So the first diagram encodes associativity of $\otimes$ and the functor η encodes the choice of a neutral element, which is why we sometimes just write $(\mathscr{C}, \otimes, I)$.

Example 4.8. Let U, V, W be finite dimensional vector spaces. Then $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ by the isomorphism sending $u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w$. Note that this is not a *strict* monoidal category since the tensor product is only associative up to a natural isomorphism and tensoring with the ground field is neutral up to a natural isomorphism.

[9] Here $\mathbb{1}$ denotes the category with a single object (and its identity arrow). π_1 and π_2 are the cannonical projection functors onto $\mathscr{C}$ and $id_{\mathscr{C}}$ is the identity functor on $\mathscr{C}$.

As the name suggests, the monoidal structure in these categories are similar to the structure of monoids (sets together with an associative binary operator and a neutral element). In a similar fashion as we define homomorphisms of monoids, we would like to define a special kind of functor that preserves the monoidal structure. This leads to the following natural definition.

Definition 4.9. Let $(\mathscr{C}, \otimes_{\mathscr{C}}, \eta_{\mathscr{C}})$ and $(\mathscr{D}, \otimes_{\mathscr{D}}, \eta_{\mathscr{D}})$ be strict monoidal categories. A *strict monoidal functor* $F : \mathscr{C} \to \mathscr{D}$ is a functor that makes the following two diagrams commute:

$$\begin{array}{ccc} \mathscr{C} \times \mathscr{C} & \xrightarrow{F\times F} & \mathscr{D} \times \mathscr{D} \\ \downarrow{\scriptstyle \otimes_{\mathscr{C}}} & & \downarrow{\scriptstyle \otimes_{\mathscr{D}}} \\ \mathscr{C} & \xrightarrow{F} & \mathscr{D} \end{array} \qquad \begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ {\scriptstyle \eta_{\mathscr{C}}}\uparrow & & \uparrow{\scriptstyle \eta_{\mathscr{D}}} \\ \mathbb{1} & \xrightarrow{id_{\mathbb{1}}} & \mathbb{1} \end{array}$$

Example 4.10. The singular homology functor

$$H_q(-) : (\mathbf{Top}, \sqcup, \varnothing) \to (\mathbf{AbGr}, \oplus, \mathbf{0})$$

is a monoidal functor since

$$H_q\left(\bigsqcup_{i=1}^{n} X_i\right) \cong \bigoplus_{i=1}^{n} H_q(X_i) \tag{2}$$

and

$$H_q(\varnothing \amalg X) \cong (\mathbf{0}) \oplus H_q(X) \cong H_q(X) \tag{3}$$

However, it is not strict, as we do not have equalities in (2) and (3), only isomorphisms.

4.3 Symmetric monoidal categories

In the category $(\mathrm{Vect}_{\mathbb{k}}, \otimes, \mathbb{k})$, for two vector spaces V and W we have that

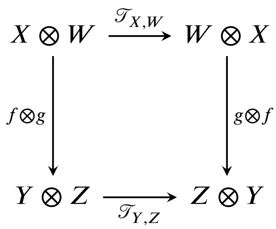
$$V \otimes W \cong W \otimes V,$$

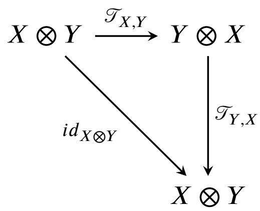
which suggests some symmetric structure in this category (with respect to the tensor product). This is true for all examples of monoidal categories we have seen so far. This is what motivates the following definition.

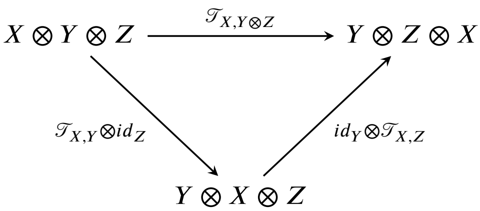
Definition 4.11. Let $(\mathscr{C}, \otimes, \eta)$ be a (strict) monoidal category. We call $\mathscr{C}$ a *(strict) symmetric monoidal category* if for every pair $X, Y \in \mathscr{C}$ there exists a *twist map*

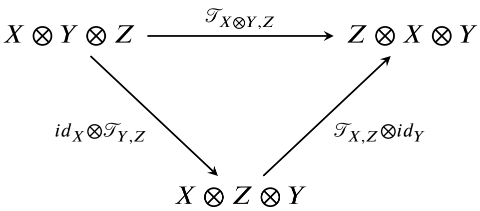
$$\mathscr{T}_{X,Y} : X \otimes Y \to Y \otimes X$$

such that for every $W, X, Y, Z \in \mathscr{C}$, $f \in Hom_{\mathscr{C}}(X, Y)$ and $g \in Hom_{\mathscr{C}}(W, Z)$ the following diagrams commute:







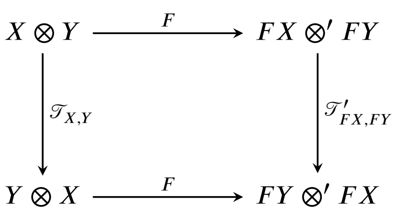


Example 4.12. In the category $(\mathbf{Set}, \times, \{\star\})$ we have the canonical twist map $X \times Y \to Y \times X$ given by

$$(x, y) \mapsto (y, x)$$

It is now natural to ask for a functor that preserves this additional structure.

Definition 4.13. Given two symmetric monoidal categories $(\mathscr{C}, \otimes, I, \mathscr{T})$ and $(\mathscr{D}, \otimes', I', \mathscr{T}')$ a monoidal functor $F : \mathscr{C} \to \mathscr{D}$ is *symmetric monoidal* if for all objects $X, Y \in \mathscr{C}$ the following diagram



commutes. In other words: the twist map in $\mathscr{C}$ is sent to the twist map in $\mathscr{D}$ under F.

Definition 4.14. Let $F, G : \mathscr{C} \to \mathscr{D}$ be monoidal functors between the monoidal categories $(\mathscr{C}, \otimes)$ and $(\mathscr{D}, \otimes')$. A natural transformation $\alpha : F \Rightarrow G$ is said to be *monoidal* if for every pair of objects $X, Y \in \mathscr{C}$ the following diagrams commute:

$$
\begin{array}{ccc}
FX \otimes' FY & \xrightarrow{\alpha_X \otimes' \alpha_Y} & GX \otimes' GY \\
\| & & \| \\
F(X \otimes Y) & \xrightarrow{\alpha_{X \otimes Y}} & G(X \otimes Y)
\end{array}
\qquad
\begin{array}{ccc}
I_{\mathscr{D}} & & \\
\| & \searrow\!\!\!\!= & \\
F(I_{\mathscr{C}}) & \xrightarrow{\alpha_{I_{\mathscr{D}}}} & G(I_{\mathscr{C}})
\end{array}
$$

Remark 4.15. Monoidal natural transformations automatically preserves the symmetric structure. What we mean by this is the following: Let $(\mathscr{C}, \amalg, I, \mathscr{T})$ and $(\mathscr{D}, \otimes, J, \sigma)$ be symmetric monoidal categories. Let F and G be symmetric monoidal functors from $\mathscr{C}$ to $\mathscr{D}$, and let $\alpha : F \Rightarrow G$ be a monoidal natural transformation. Then, the following diagram commutes:

$$
\begin{array}{ccc}
F(Y \amalg X) & \xrightarrow{\alpha_{Y \amalg X}} & G(Y \amalg X) \\
{\scriptstyle F(\mathscr{T}_{X,Y})}\downarrow & & \downarrow{\scriptstyle G(\mathscr{T}_{X,Y})} \\
F(X \amalg Y) & \xrightarrow{\alpha_{X \amalg Y}} & G(X \amalg Y)
\end{array}
$$

To see this, observe that this is exactly the outer square in the following diagram:

$$
\begin{array}{ccc}
F(X \amalg Y) & \xrightarrow{\alpha_{X \amalg Y}} & G(X \amalg Y) \\
\| & 1 & \| \\
FX \otimes FY & \xrightarrow{\alpha_X \otimes \alpha_Y} & GX \otimes GY \\
{\scriptstyle \sigma_{FX,FY}}\downarrow{\scriptstyle \cong} & 2 & {\scriptstyle \cong}\downarrow{\scriptstyle \sigma_{GX,GY}} \\
FY \otimes FX & \xrightarrow{\alpha_Y \otimes \alpha_X} & GY \otimes GX \\
\| & 3 & \| \\
F(Y \amalg X) & \xrightarrow{\alpha_{Y \amalg X}} & G(Y \amalg X)
\end{array}
$$

The upper and lower squares (1 and 3) commutes because α is monoidal. The middle square (2) commutes because of the naturality of the twist map σ. Thus, the outer square commutes as well.

Now we want to introduce an important concept: What does it mean for two categories to be "equal"? The following definition might seem the most natural:

Definition 4.16. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. We say that $\mathscr{C}$ and $\mathscr{D}$ are *isomorphic*, denoted $\mathscr{C} \cong \mathscr{D}$, if there exists a functor $F : \mathscr{C} \to \mathscr{D}$ and a functor $G : \mathscr{D} \to \mathscr{C}$ such that $F \circ G$ and $G \circ F$ are the identity functors on $\mathscr{D}$ and $\mathscr{C}$, respectively.

We will use this definition of isomorphic categories later in this text. However, it turns out that most of the time this requirement is too strong. That is, there are categories we would like to be "equal" that does not have such an "invertible" functor between them. Therefore, we relax this condition a bit and give the following defintion:

Definition 4.17. Given two categories $\mathscr{C}$ and $\mathscr{D}$ we say they are *equivalent*, denoted $\mathscr{C} \simeq \mathscr{D}$, if there exists functors

$$F : \mathscr{C} \to \mathscr{D} \quad \text{and} \quad G : \mathscr{D} \to \mathscr{C}$$

and natural isomorphisms

$$\eta : F \circ G \Rightarrow \mathrm{id}_{\mathscr{D}} \quad \text{and} \quad \alpha : \mathrm{id}_{\mathscr{C}} \Rightarrow G \circ F.$$

Using the axiom of (global) choice, one can show that this is equivalent to the following definition[10]:

Definition 4.18. Given two categories $\mathscr{C}$ and $\mathscr{D}$ we say they are *equivalent*, denoted $\mathscr{C} \simeq \mathscr{D}$, if there exists a functor

$$F : \mathscr{C} \to \mathscr{D}$$

which is fully faithful and essentially surjective (dense).

Let us also define the concept of being (symmetric) monoidally equivalent:

Definition 4.19. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories which are equivalent in terms of definition 4.17. If the functors F and G are monoidal and the natural transformations η and α are monoidal, then we say that $\mathscr{C}$ and $\mathscr{D}$ are *monoidally equivalent*. If $\mathscr{C}$ and $\mathscr{D}$ are symmetric monoidal categories and the functors F and G are symmetric monoidal, then we say that $\mathscr{C}$ and $\mathscr{D}$ are *symmetric monoidally equivalent*.

Note that when the categories in question are (symmetric) monoidal, we often take "equivalent" to mean (symmetric) monoidally equivalent.

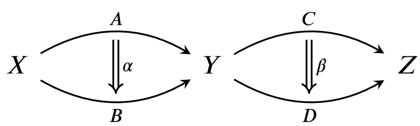
4.4 Some useful lemmas

We now prove some useful lemmas that we will need later.

Lemma 4.20. Equivalence of categories is transitive. That is, if we have that $A \simeq B$ and $B \simeq C$, then $A \simeq C$

Proof. Given essentially surjective and fully faithful functors $F : A \to B$ and $G : B \to C$, consider the composition $G \circ F : A \to C$. Given $c \in C$, there exists a $b \in B$ so that $Gb \cong c$ and $a \in A$ so that $Fa \cong b$. And hence $G \circ Fa \cong c$ which proves that $G \circ F$ is essentially surjective. Since the composition of bijections are again a bijection, the composition is fully faithful as well. □

Given two natural transformations

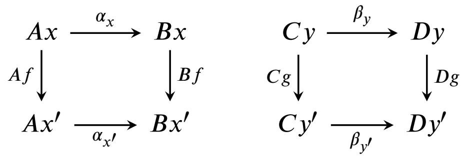


we want to define the *horizontal composition* $\beta * \alpha : CA \Rightarrow DB$.[11] Let $f : x \to x'$

[10] A proof of this is found in [Rie17, p. 31].

[11] Sometimes, to save space, we simply write GF instead of $G \circ F$.

and $g : y \to y'$ be morphisms in X and Y, respectively. From the naturality of α and β we have the following two commuting diagrams:



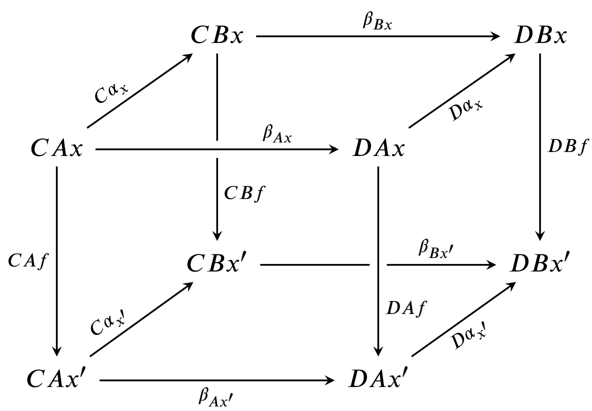
Now, we have the following two candidates for $(\beta * \alpha)_x$:

$$CAx \xrightarrow{\beta_{Ax}} DAx \xrightarrow{D\alpha_x} DBx \quad \text{and} \quad CAx \xrightarrow{C\alpha_x} CBx \xrightarrow{\beta_{Bx}} DBx.$$

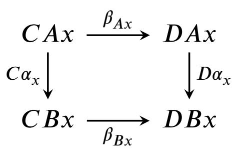
That is, $D\alpha_x \circ \beta_{Ax}$ and $\beta_{Bx} \circ C\alpha_x$. We now prove that these are the same.

Lemma 4.21. (Let the setup be as above.) We have that $D\alpha_x \circ \beta_{Ax} = \beta_{Bx} \circ C\alpha_x$.

Proof. Consider the following commutative cube:



The commutativity of this cube expresses the naturality of both candidates. The "top face" which is the following square:



expresses the naturality of β, which is to say that the two possible choices for $\beta * \alpha$ are the same. □

In the following lemma and its proof we will use the notation $[A, C]$ as a shorthand for the functor category $\mathrm{Fun}(A, C)$.

Lemma 4.22. Let A, B and C be categories and suppose $A \simeq B$. Then we have that $[A, C] \simeq [B, C]$.

Proof. Since $A \simeq B$, we have functors $F : A \to B$ and $G : B \to A$ together with two natural isomorphisms

$$\eta : F\circ G \overset{\sim}{\Rightarrow} \mathrm{id}_B \quad \text{and} \quad \epsilon : \mathrm{id}_A \overset{\sim}{\Rightarrow} G\circ F.$$

The functors F and G induce functors on the functor categories. On objects they are given as

$$\begin{aligned} F^* : [B, C] &\to [A, C] \\ \Phi &\mapsto \Phi\circ F\,, \text{ and} \\ G^* : [A, C] &\to [B, C] \\ \Psi &\mapsto \Psi\circ G. \end{aligned}$$

Let us define F^* on morphisms as follows: given a natural transformation $\alpha : \Phi_1 \Rightarrow \Phi_2$, where $\Phi_1, \Phi_2 \in [B, C]$, we define the component of $F^*\alpha$ at $a \in A$ as

$$(F^*\alpha)_a := \alpha_{F(a)} : \Phi_1(F(a)) \to \Phi_2(F(a)).$$

Similarly, for G^*, given a morphism $\beta : \Psi_1 \Rightarrow \Psi_2$ in $[A, C]$ we define

$$(G^*\beta)_b := \beta_{G(b)} : \Psi_1(G(b)) \to \Psi_2(G(b)).$$

We now define $\epsilon^* : \mathrm{id}_{[A,C]} \overset{\sim}{\Rightarrow} F^*\circ G^*$ on components as

$$\epsilon^*(\Psi) : \Psi \Rightarrow \Psi\circ G\circ F,$$

where we define the component of $\epsilon^*(\Psi)$ at $a \in A$ to be

$$(\epsilon^*(\Psi))_a := \Psi(\epsilon_a) : \Psi(a) \xrightarrow{\cong} \Psi(G\circ F(a)).$$

This is an isomorphism because $\epsilon_a : a \to G\circ F(a)$ is an isomorphism by the equivalence assumption.

Applying Ψ to the following diagram of ϵ:

$$\begin{array}{ccc} a & \xrightarrow{\epsilon_a} & G\circ F(a) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G\circ F(g)} \\ a' & \xrightarrow[\epsilon_{a'}]{} & G\circ F(a') \end{array}$$

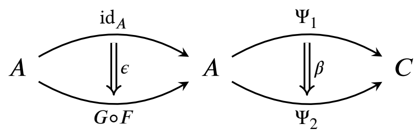
we obtain this diagram:

$$\begin{array}{ccc} \Psi(a) & \xrightarrow{\Psi(\epsilon_a)} & \Psi(G\circ F(a)) \\ {\scriptstyle \Psi(g)}\downarrow & & \downarrow{\scriptstyle \Psi(G\circ F(g))} \\ \Psi(a') & \xrightarrow[\Psi(\epsilon_{a'})]{} & \Psi(G\circ F(a')) \end{array}$$

which shows that $\epsilon^*(\Psi)$ is in fact natural. We now show that ϵ^* is a natural isomorphism. (I.e., it is natural in Ψ.) That is the same as commutativity of the diagram:

$$\begin{array}{ccccc} \Psi_1 & \xRightarrow[\cong]{\epsilon^*(\Psi_1)} & F^*\circ G^*(\Psi_1) & \overset{\text{by def.}}{=\!=\!=} & \Psi_1\circ G\circ F \\ \beta\Downarrow & & \Downarrow F^*\circ G^*(\beta) & & \\ \Psi_2 & \xRightarrow[\epsilon^*(\Psi_2)]{\cong} & F^*\circ G^*(\Psi_2) & =\!=\!= & \Psi_2\circ G\circ F \end{array}$$

Consider the horizontal composition of the following natural transformations:



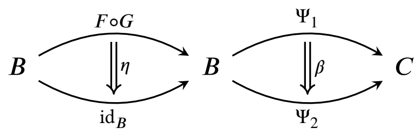
By lemma 4.21, this yields the equation

$$\Psi_2(\epsilon_a)\circ\beta_a = \beta_{G\circ F(a)}\circ\Psi_1(\epsilon_a)$$

which is by definition the same as

$$\epsilon^*(\Psi_2)_a\circ\beta_a = (F^*\circ G^*(\beta))_a\circ\epsilon^*(\Psi_1)_a$$

Hence, ϵ^* is in fact a natural. Similarly, by considering the composition



we have that η^* is a natural isomorphism as well. □

Definition 4.23. A *skeleton* of a category $\mathscr{C}$ is a full subcategory S of $\mathscr{C}$ which has exactly one object from each isomorphism class in $\mathscr{C}$. By full, we mean that the inclusion functor $S \hookrightarrow \mathscr{C}$ is full.

Note that the inclusion functor $S \hookrightarrow \mathscr{C}$ is essentially surjective by definition and restricts to the identity on sets of morphisms[12]. Thus, we have the following lemma:

Lemma 4.24. If $\mathscr{C}$ is a category and S is a skeleton of $\mathscr{C}$, then $C \simeq \mathscr{C}$.

In particular, lemma 4.22 implies the following corollary:

Corollary 4.25. If S is a skeleton of $\mathscr{C}$ then we have that

$$\mathrm{Fun}(S, \mathscr{D}) \simeq \mathrm{Fun}(\mathscr{C}, \mathscr{D}).$$

[12]Every inclusion functor is faithful since it restricts to the inclusion on sets of morphisms.

5 Cobordisms and TQFTs

Our main goal of this section is to define "oriented cobordisms" and to define a cobordism category. This category will consist of closed $(n-1)$-dimensional manifolds as its objects, and n-dimensional manifolds with boundary as its morphisms. We require all manifolds to be orientable. We shall denote this category $\mathbf{nCob^{or}}$. We are also going to define the required constructions such as identity morphisms and composition in $\mathbf{nCob^{or}}$. For the purposes of this text, we are mainly going to be interested in the case when $n = 2$. That is, the category where the morphisms are two-dimensional oriented cobordisms.

5.1 Oriented cobordisms

Given a disjoint union of two manifolds M and N, we can ask if there exists a manifold W of one dimension higher that has these two manifolds as its boundary. If that is the case, we say that M and N are cobordant and that W is a cobordism between M and N. We will use our concept of incoming and outgoing boundaries to define what is known as oriented cobordisms. We can think of these as cobordisms with a direction. We need this because oriented cobordisms are going to be the arrows in the cobordism category. Note that we will from now on by "manifold" mean a compact smooth manifold.

If we let Σ_n denote the disjoint union of n copies of the circle. Then, one possible cobordism between Σ_2 and Σ_3 is the following one:

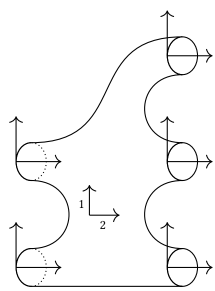


Figure 5.1.1: An oriented cobordism.

Here comes two important remarks before we give the precise definition of an oriented cobordism:

Remark 5.1.

1. Recall that by convention we draw the in-boundary on the left and the out-boundary on the right. We will therefore usually omit the arrows indicating direction.

2. It is not enough to simply give an oriented manifold with boundary. To tell which boundary components are incoming and which are outgoing, we also have to specify the orientation of the boundary components.

Definition 5.2. Given two closed and oriented $(n-1)$-manifolds Σ and Σ', an *oriented cobordism* from Σ to Σ' is an oriented n-manifold M with boundary, together with two orientation preserving diffeomorphisms from Σ and Σ' onto the in-boundary and the out-boundary of M, respectively.

$$\Sigma \xrightarrow[f_{\text{in}}]{\cong} \partial_{\text{in}} M \subset M \supset \partial_{\text{out}} M \xleftarrow[f_{\text{out}}]{\cong} \Sigma'$$

We denote such a cobordism by $M : \Sigma \Rightarrow \Sigma'$.

We explain the motivation for defining it with the maps f_{in} and f_{out} like this in the following remark:

Remark 5.3. A more naive definition could be as follows: an oriented cobordism from Σ to Σ' is a manifold with Σ as its in-boundary and Σ' as its out-boundary. The problem with this definition is that we can not have a cobordism from Σ to itself, because Σ can not be both the in-boundary and the out-boundary simultaneously. That is the reason we just require the boundaries to be embedded in M. For example, the identity cobordism on Σ,

$$\text{id}_{\Sigma} : \Sigma \Rightarrow \Sigma,$$

is something we need for our cobordism category. We shall soon see that the identity cobordism id_{Σ} is exactly the (class of the) cylinder over Σ.

5.1.1 Cobordisms induced by diffeomorphisms

Remark 5.4. Let Σ be an oriented manifold without boundary, and let $\{v_1, \dots, v_n\}$ be a positive basis of $T_x\Sigma$ for some $x \in \Sigma$. Let $[0, 1]$ be oriented with $\{e_1\}$ as a positive basis. Following the convention of definition 2.22, the induced orientation on $\Sigma \times [0, 1]$ is given by declaring $\{v_1, \dots, v_n, e_1\}$ a positive basis.

Given an orientation preserving diffeomorphism $\phi : \Sigma \to \Sigma'$ between two $(n-1)$-manifolds, we can define an induced cobordism $M : \Sigma \Rightarrow \Sigma'$ in the following way: Let $M := \Sigma' \times [0, 1]$ be given the induced product orientation described in remark 5.4. We have a natural embedding

$$\Sigma' \xrightarrow{\cong} \Sigma' \times \{1\} = \partial_{\text{out}} M$$

defined by $x \mapsto (x, 1)$. We define the other diffeomorphism by the following composition:

$$\Sigma \xrightarrow{\phi} \Sigma' \xrightarrow{\cong} \Sigma' \times \{0\} = \partial_{\text{in}} M.$$

This proves the following lemma:

Lemma 5.5. Diffeomorphic manifolds are cobordant.

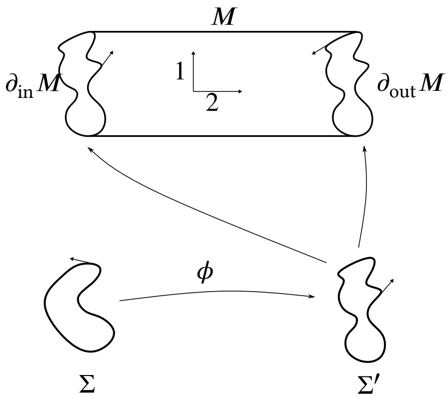


Figure 5.1.2: An oriented cobordism induced from a diffeomorphism.

5.1.2 A short remark on orientations

Let S^1_+ and S^1_- be two copies of the circle with different orientations. It is not known, a priori, that there exists an orientation preserving[13] diffeomorphism between the two circles. For example, the identity map $S^1_+ \to S^1_-$ is orientation reversing. (We have to multiply by -1 so it has negative determinant.) However, the following reflection map is an orientation preserving diffeomorphism:

$$r: S^1_+ \to S^1_-$$
$$(x, y) \mapsto (-x, y)$$

It takes a positive basis of S^1_+ to a positive basis of S^1_-. The following picture gives the intuition:

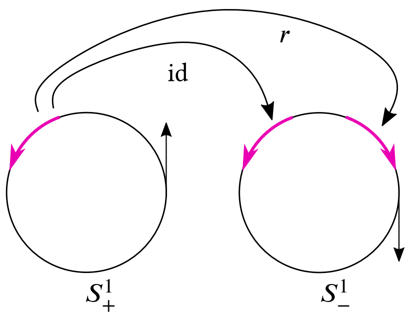


Figure 5.1.3: Orientation preserving and orientation reversing diffeomorphisms.

Another thing to note is the following: if we are given an oriented cobordism M and we reverse the orientation on M (and leave the orientation on the boundary as it is). Then, the in-boundary becomes the out-boundary and vice versa. This is because

[13]Sometimes, an orientation preserving diffeomorphism is just called an "oriented diffeomorphism".

on each boundary component, we now have to choose the other normal vector. That is, if it originally pointed inwards, we now have to pick the outward pointing normal vector to extend the basis of the tangent space. And similarly for outgoing boundary components.

For example, the pair of pants becomes the reversed pair of pants when we reverse the orientation on the cobordism:

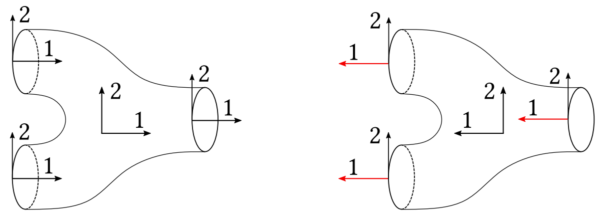
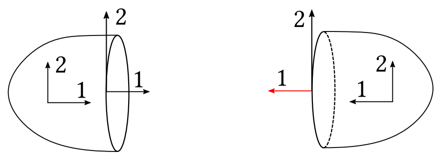


Figure 5.1.4: A new pair of pants.

Similarly, but more dramatically, the "birth of a circle" becomes the "death of a circle":



The important thing to take away from this and to keep in mind is that orientations are always lurking behind the scenes.

5.1.3 Some useful cobordisms

If we take the identity map on Σ, we get an induced cobordism from Σ to itself. We call this cobordism the *identity* on Σ for reasons that will become clear when we make everything into a category. In $\mathbf{2Cob}^{\mathbf{or}}$ we can picture the cylinder, which is the identity cobordism on Σ_n (the disjoint union of n circles) as follows:

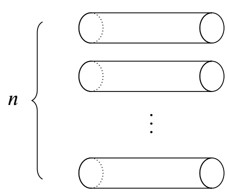


Figure 5.1.5: Identity cobordism on Σ_n.

Given a disjoint union of two $(n-1)$-manifolds X and Y we can interchange the factors, so there is a natural oriented diffeomorphism

$$T_{X,Y} : X \amalg Y \xrightarrow{\cong} Y \amalg X .$$

Notice that $T_{Y,X}$ is the corresponding inverse map. Let $\mathscr{T}_{X,Y}$ denote the twist cobordism induced by $T_{X,Y}$. In $\mathbf{2Cob^{or}}$ we can picture it like this:

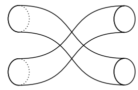


Figure 5.1.6: The twist cobordism induced from $T_{X,Y}$.

Taking the composition (which we shall define later in this section) $\mathscr{T}_{Y,X} \circ \mathscr{T}_{X,Y}$ we have in $\mathbf{2Cob^{or}}$ the following picture:

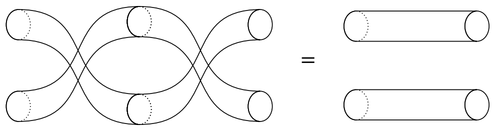


Figure 5.1.7: The twist cobordism is an invertible cobordism.

Remark 5.6. This cobordism and the relation $\mathscr{T}_{X,Y} \circ \mathscr{T}_{Y,X} = \mathrm{id}_{X \amalg Y}$ are exactly what is going to give the symmetric structure on $\mathbf{2Cob^{or}}$.

Now, let Σ be an oriented closed $(n-1)$-dimensional manifold, and let M be the cylinder $\Sigma \times [0,1]$. We can reverse the orientation on the out-boundary in the following way: at each point $x \in \Sigma \times \{1\}$ we have a positive basis $\{v_1, \dots, v_{n-1}\}$ for $T_x(\Sigma \times \{1\})$. Define a new orientation on $\Sigma \times \{1\}$ by setting $\{-v_1, v_2, \dots, v_{n-1}\}$ as a positive basis. This turns $\Sigma \times \{1\}$ into an in-boundary and gives a cobordism $M : \Sigma \amalg \overline{\Sigma} \Rightarrow \varnothing$.

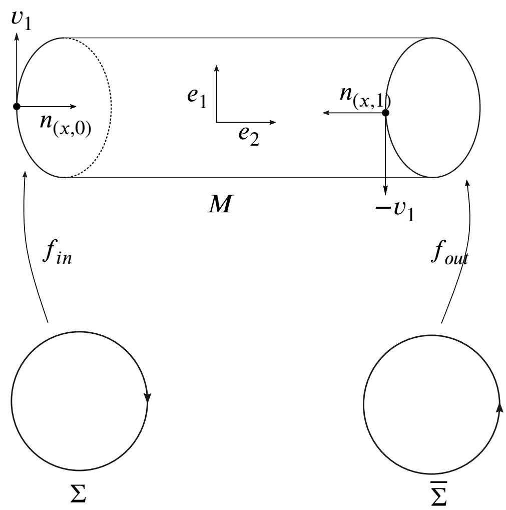


Figure 5.1.8: Reversing the orientation on the out-boundary.

Using our convention with in-boundaries on the left we picture this cobordism as in figure 5.1.9 below.

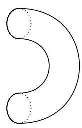


Figure 5.1.9

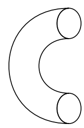


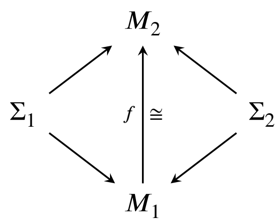
Figure 5.1.10

If we instead change the in-boundary of the cylinder into an out-boundary by reversing its orientation we get the mirror image depicted in figure 5.1.10. This is a cobordism $\emptyset \Rightarrow \overline{\Sigma} \amalg \Sigma$. These two cobordisms are often called *macaronis*, *elbows* or *U-tubes*.

5.2 A notion of equivalence of cobordisms

To be able to define composition of cobordisms we will first need to define a notion of equivalence. In addition to requiring the cobordisms to be diffeomorphic, we want the diffeomorphism to preserve orientation and respect the boundaries. The precise definition is as follows:

Definition 5.7. Let M_1 and M_2 be two cobordisms, both from Σ_1 to Σ_2. We say M_1 and M_2 are *equivalent* if there exists an orientation preserving diffeomorphism $f : M_1 \to M_2$ such that the following diagram commutes:



For reasons that will become apparent we will use this notion of equivalence to define *cobordism classes* as the morphism in $\mathbf{nCob^{or}}$.

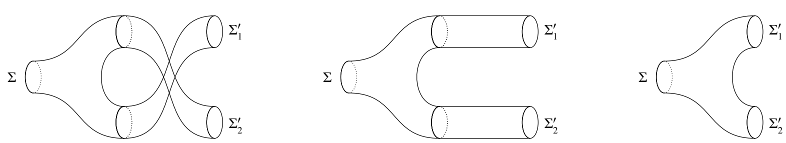


Figure 5.2.1: Example of equivalent cobordisms.

Remark 5.8. Now here are two examples of cobordisms that are not equivalent even though they have the same number of boundary components.

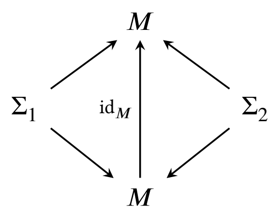
The two on the left are certainly diffeomorphic as manifolds. However, as oriented cobordism, they are not equivalent. This is because we require that the diffeomorphism restricts to the identity on the boundary, and going from one to the other would require interchanging the boundary components. The two on the right are not equivalent simply because the genus differs.

Lemma 5.9. Being equivalent is an equivalence relation.

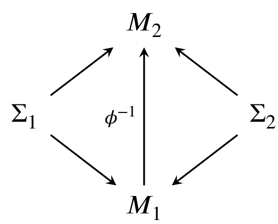
Proof. We introduce the following notation for this proof:

$$M_1 \sim M_2 \iff M_1 \text{ and } M_2 \text{ are equivalent.}$$

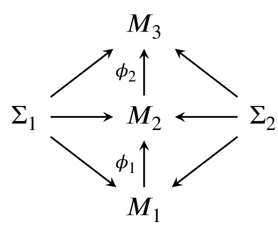
1. *Reflexivity:* Take the identity on M to obtain the following diagram:



2. *Symmetry:* Suppose $M_1 \sim M_2$ via the diffeomorphism ϕ. We then take ϕ^{-1} to obtain the following diagram:



3. *Transitivity:* Suppose $M_1 \sim M_2$ and $M_2 \sim M_3$. We then take the composition $\phi_2 \circ \phi_1$ in the following diagram:



□

Note that from now on we will by "(oriented) cobordisms" mean the equivalence classes of oriented cobordisms. So that when we write M, we actually mean the cobordism class M represents.

Now, we want to be able to take the disjoint union of cobordisms. We define this in a natural way:

Definition 5.10. Let $M : \Sigma_0 \Rightarrow \Sigma_1$ and $M' : \Sigma_0' \Rightarrow \Sigma_1'$ be two oriented cobordisms. Then we define the disjoint union of M and M', denoted $M \amalg M'$, by the cobordism class represented by $M_r \amalg M_r'$ [14] where M_r and M_r' are representatives of M and M', respectively. In other words, we have the following induced diagram:

$$\Sigma_0 \amalg \Sigma_0' \longrightarrow M \amalg M' \longleftarrow \Sigma_1 \amalg \Sigma_1'$$

We should note that the disjoint union of cobordisms "respects" the ordering of the boundary components. For example, consider the figure on the left in remark 5.8: the twist cobordism is *not* the disjoint union of two cylinders. However, the cylinders in the same figure certainly are the disjoint union of two cylinders. Another example is the following two cobordisms:

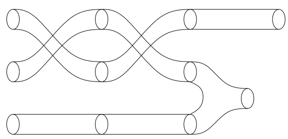


Figure 5.2.3: This is a disjoint union of cobordisms.

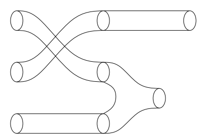


Figure 5.2.4: This is not.

Note that we also have the empty cobordism $\emptyset_n : \emptyset_{n-1} \Rightarrow \emptyset_{n-1}$.

5.3 Gluing of cobordisms

To create a category where morphisms are cobordisms we need a well-defined notion of composition. We will realize this by gluing cobordisms along a common piece of boundary. The resulting manifold should be a cobordism again so it should be a smooth manifold. And since we now have moved to cobordism classes, we have to make sure that choice of representative does not matter. The idea is something like this: first we prove that our cobordisms look like cylinders near the boundary. Then we show how we can glue cylinders together and hence how we can glue any cobordisms along a common boundary component.

[14] Note that this $\amalg$ is just taking disjoint union of manifolds and is not the same as the $\amalg$ for cobordism (classes) that we are now defining.

5.3.1 Gluing topological spaces

We begin by describing a procedure for gluing topological spaces. Given two continuous injections

$$M \xleftarrow{f} \Sigma \xrightarrow{g} N$$

We define the gluing of M and N along Σ as follows:

$$M \amalg_\Sigma N := {M \amalg N}/{\sim}$$

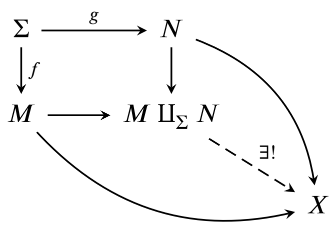
where for $x \in M$ and $y \in N$, $x \sim y \iff \exists z \in \Sigma$ such that $f(z) = x$ and $g(z) = y$.

We define a subset U of $M \amalg_\Sigma N$ to be open if both $f^{-1}(U)$ and $g^{-1}(U)$ are open in M and N respectively.

Remark 5.11. $M \amalg_\Sigma N$ is the pushout of $M \xleftarrow{f} \Sigma \xrightarrow{g} N$ in **Top**. That is, for every commutative diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{g} & N \\ \downarrow{\scriptstyle f} & & \downarrow \\ M & \longrightarrow & X \end{array}$$

there is a unique continuous map $M \amalg_\Sigma N \to X$ such that the following diagram commutes:



5.3.2 Gluing C^0-manifolds

We now move on to gluing together topological manifolds M and N along a common boundary component Σ and worry about the smooth structure later. Again, we start with two maps from Σ to each of the boundary components we want to glue along:

$$M \xleftarrow{f} \Sigma \xrightarrow{g} N$$

For points in $M \amalg_\Sigma N$ that do not lie in the image of either map we already have charts. So what needs to be done is to construct charts about every point where we have glued the two manifolds together. Take an open set U around such a point. Let $U_M = U \cap M$ and $U_N = U \cap N$, which are open by definition, and shrink U until we have charts $U_N \to \mathbb{H}^n$ and $U_M \to \mathbb{H}^n$. Since we are working with maximal atlases, we can choose charts $\phi_M : U_M \to \mathbb{R}^n_+$ and $\phi_N : U_N \to \mathbb{R}^n_-$.[15]

We have that $U = U_M \amalg_{\Sigma \cap U} U_N$, so the universal property mentioned in remark 5.11 gives us the dashed arrow ϕ in this diagram:

[15] We define $\mathbb{R}^n_+ := \mathbb{H}^n = \{(x_1, \dots, x_n) | x_n \geq 0\}$ and $\mathbb{R}^n_- := \{(x_1, \dots, x_n) | x_n \leq 0\}$.

$$\begin{array}{ccccc} & \swarrow & \Sigma \cap U & \searrow & \\ U_M & \longrightarrow & U & \longleftarrow & U_N \\ \downarrow{\scriptstyle \phi_M} & & {\scriptstyle \phi}\,\vdots\,{\scriptstyle \exists!} & & \downarrow{\scriptstyle \phi_N} \\ \mathbb{R}^n_+ & \longrightarrow & \mathbb{R}^n & \longleftarrow & \mathbb{R}^n_- \end{array}$$

Similarly, let V be the gluing of $\phi_M(U_M)$ and $\phi_N(U_N)$ along their common boundary in $\mathbb{R}^{n-1}$, and observe that $\mathbb{R}^n = \mathbb{R}^n_+ \amalg_{\mathbb{R}^{n-1}} \mathbb{R}^n_-$. We apply remark 5.11 to get the inverse map $V \xrightarrow{\cong} U$.

$$\begin{array}{ccccc} U_M & \longrightarrow & U & \longleftarrow & U_N \\ {\scriptstyle \phi_M^{-1}}\uparrow & & {\scriptstyle \exists!}\,\vdots & & \uparrow{\scriptstyle \phi_N^{-1}} \\ \phi_M(U_M) & \longrightarrow & V & \longleftarrow & \phi_N(U_N) \\ & \nwarrow & V \cap \mathbb{R}^{n-1} & \nearrow & \end{array}$$

That this is the inverse map follows from universiality.

Note that we made a choice of the charts ϕ_M and ϕ_N. Different choices could have given us a different ϕ. It is shown in [Koc04, p. 38] that the different charts we get from this construction belong to the same maximal atlas. Hence, the C^0 structure on $M \amalg_\Sigma N$ constructed in this way is well-defined.

Here is a (hopefully illuminating) illustration of how this gluing process might look like when we picture the manifolds in $\mathbb{R}^3$:

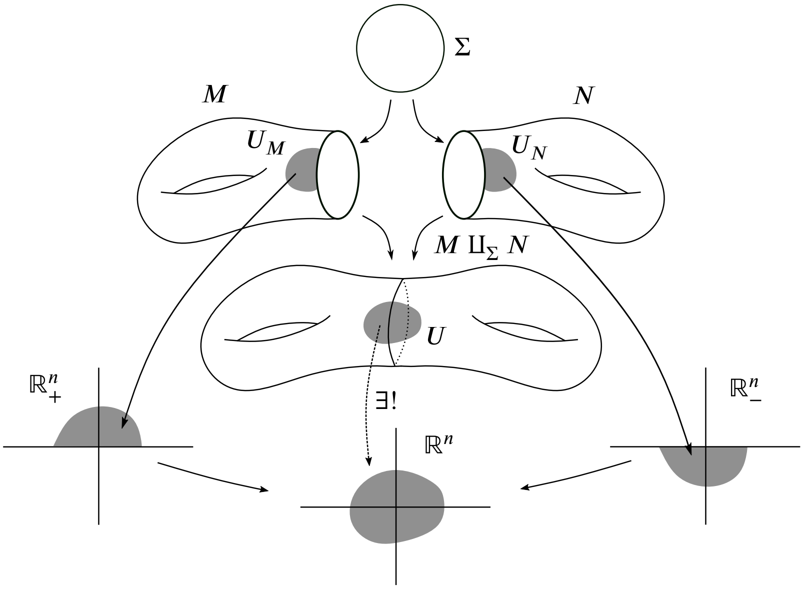


Figure 5.3.1: Gluing of two manifolds.

5.3.3 Gluing smooth manifolds

Unfortunately, the gluing procedure we described for C^0-manifolds does not work out as nicely for smooth manifolds, as we shall see in the following example:

Example 5.12. Let $M_0 := [-1, 0]$, $M_1 := [0, 1]$ and let M be the gluing of M_0 and M_1 at the common boundary $\{0\}$. That is, $M = [-1, 1]$. For some $\epsilon > 0$, let $U_0 = (-\epsilon, 0]$ and $U_1 = [0, \epsilon)$ be open sets around 0 in M_0 and M_1, respectively. Choose the charts $\phi_0 : U_0 \to \mathbb{R}_-$ and $\phi_1 : U_1 \to \mathbb{R}_+$, both equal to the inclusion map $x \mapsto x$. Now, glue these maps to obtain a chart around $0 \in M$ as follows:

$$\psi : (-\epsilon, \epsilon) \to \mathbb{R}$$
$$x \mapsto \begin{cases} \phi_0(x) & \text{if } x \in U_0 \\ \phi_1(x) & \text{if } x \in U_1. \end{cases}$$

We can choose another chart around $0 \in M_1$: let $\phi_1' : U_1 \to \mathbb{R}_+$ be defined by $x \mapsto x^2$. (With smooth inverse $x \mapsto \sqrt{x}$.) We now glue ϕ_0 and ϕ_1' to obtain the following chart around $0 \in M$:

$$\psi' : (-\epsilon, \epsilon) \to \mathbb{R}$$
$$x \mapsto \begin{cases} \phi_0(x) & \text{if } x \in U_0 \\ \phi_1'(x) & \text{if } x \in U_1. \end{cases}$$

In total, we have constructed two different charts around $0 \in M$. However, the transistion map $\psi' \circ \psi^{-1}$,

$$x \mapsto \begin{cases} x & \text{if } x \in U_0 \\ x^2 & \text{if } x \in U_1, \end{cases}$$

is not smooth at $x = 0$. This means that the two different charts we have constructed around $0 \in M$ belongs to two different maximal atlases. However, they are diffeormorphic: Let $\mathcal{A}_\psi$ and $\mathcal{A}_{\psi'}$ be the smooth structures on M induced by the atlases containing ψ and ψ', respectively. Since $(M, \mathcal{A}_\psi)$ and $(M, \mathcal{A}_{\psi'})$ both are connected 1-dimensional manifolds with boundary, we know that they have to be diffeomorphic.

The following theorem is called the *regular interval theorem*. We state it as in [Koc04, p. 41] and omit the proof. A proof of the theorem can be found in [Hir97, p. 153].

Theorem 5.13. Let $M : \Sigma_0 \Rightarrow \Sigma_1$ be a cobordism and let $f : M \to [0,1]$ be a smooth map with no critical points such that the inverse images of 0 and 1 are Σ_0 and Σ_1 respectively. Then there exists diffeomorphisms $\phi_0 : \Sigma_0 \times [0,1] \xrightarrow{\cong} M$ and $\phi_1 : \Sigma_1 \times [0,1] \xrightarrow{\cong} M$ that are compatible with the projections onto the second coordinate. In other words, we have the following commutative diagram:

$$\begin{array}{ccccc} \Sigma_0 \times [0,1] & \xrightarrow[\phi_0]{\cong} & M & \xleftarrow[\phi_1]{\cong} & \Sigma_1 \times [0,1] \\ & \searrow^{\text{proj.}} & \downarrow f & \swarrow_{\text{proj.}} & \\ & & [0,1] & & \end{array}$$

We should read this as follows: if M admits such a map f with no critical points, then M is (diffeomorphic to) a cylinder.

Of course, to harvest the usefulness of this theorem, one needs to show that such an f exists. In fact, a Morse function always exists as they form a dense subset of the set of smooth maps $M \to [0,1]$. (See [Koc04, p. 17] for more details, and [Hir97, p. 147] for a proof of this claim.) The following corollary is of great importance for our gluing needs:

Corollary 5.14. [Koc04, p. 41] Let $M : \Sigma_0 \Rightarrow \Sigma_1$ be a cobordism. Then we can decompose M as $M_{[\epsilon_1,1]} \circ M_{[\epsilon_0,\epsilon_1]} \circ M_{[0,\epsilon_0]}$ such that $M_{[0,\epsilon]}$ is diffeomorphic to a cylinder over Σ_0 and $M_{[\epsilon_1,1]}$ is diffeomorphic to a cylinder over Σ_1.

That is, near the boundaries there is some collar which looks like the cylinder over that boundary. The following figure should give the intuition needed:

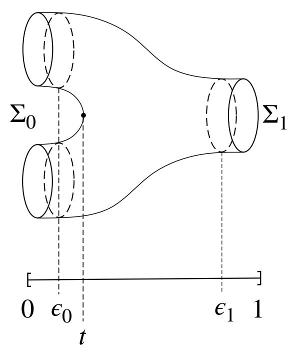


Figure 5.3.2: Collars near the boundary components.

Given two cylinders $\Sigma \times [0,1]$ and $\Sigma \times [1,2]$, we would like to glue them along $\Sigma \times \{1\}$. The idea is to construct a smooth structure on $[0,2]$ as we did in example 5.12, and get an induced smooth structure on $\Sigma \times [0,2]$. Let $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$ be cobordisms equivalent to cylinders over Σ_1. That is, we have diffeomorphisms

$$\phi_0 : M_0 \xrightarrow{\cong} \Sigma_1 \times [0,1] \text{ and } \phi_1 : M_1 \xrightarrow{\cong} \Sigma_1 \times [1,2].$$

Define

$$\phi := \phi_0 \amalg_{\Sigma_1} \phi_1 : M_0 \amalg_{\Sigma_1} M_1 \to \Sigma_1 \times [0,2],$$

which is a continuous map given by the universal property in the category of continuous maps. Since $\Sigma_1 \times [0,2]$ has a smooth structure compatible with the smooth structures on $\Sigma_1 \times [0,1]$ and $\Sigma_1 \times [1,2]$, we pull back the smooth structure along ϕ to get the desired smooth structure on $M = M_0 \amalg_{\Sigma_1} M_1$.

Now the crucial idea, when given two arbitrary cobordisms, is to take a Morse function such that we have collars near the common boundary. We can do this because of corollary 5.14. Then we glue these cylinder components as above to obtain a smooth structure on the composition. It turns out, even though the smooth structure is not canonical, that if given two different smooth structures on the composition, which agree with the original smooth structures on each components, then they are diffeomorphic. We state this as a lemma, with details given in [Koc04, p. 42].

Lemma 5.15. [Koc04, p. 42]. The smooth structure on the gluing $M_0 \amalg_{\Sigma_1} M_1$ is unique up to diffeomorphism. Also, the diffeomorphism restricts to the identity map on the boundary.

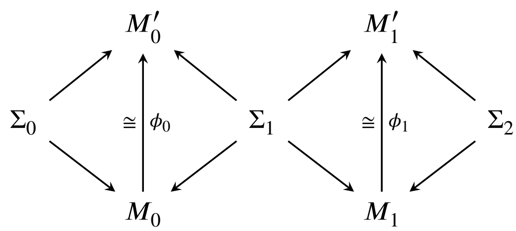
We sum up our discussion above into the following lemma:

Lemma 5.16. The gluing of two cylinders is (diffeomorphic to) a cylinder.

Definition 5.17. We define the *composition* of $M_1 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_2 : \Sigma_1 \Rightarrow \Sigma_2$, denoted $M_1 \circ M_0$, as the gluing $M_0 \amalg_{\Sigma_1} M_1$.

Lemma 5.18. Gluing of cobordisms is well defined.

Proof. We need to check that the composition is independent of the chosen representative. We take two representatives from each class and obtain the following diagram:



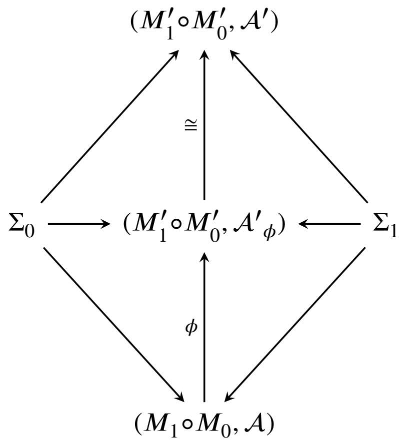
This gives rise to the two different gluings $M_1 \circ M_0$ and $M_1' \circ M_0'$ with certain smooth structures. Let us call these structures $\mathcal{A}$ and $\mathcal{A}'$, respectively. The two diffeomorphisms ϕ_0 and ϕ_1 glue in the category of continuous maps and give a homeomorphism

$$\phi : M_1 \circ M_0 \xrightarrow{\cong} M_1' \circ M_0'.$$

Now we can use ϕ to define a smooth structure $\mathcal{A}'_\phi$ on $M_1' \circ M_0'$. We define the smooth structure on $M_1' \circ M_0'$ as follows: for each chart $\psi : U \to \mathbb{H}^n$ on $M_1 \circ M_0$, define the induced chart on $M_1' \circ M_0'$ as $\psi' := \psi \circ \phi^{-1} : \phi(U) \to \mathbb{H}^n$. With this smooth structure on $M_1' \circ M_0'$,

$$\phi : (M_1 \circ M_0, \mathcal{A}) \to (M_1' \circ M_0', \mathcal{A}'_\phi)$$

is a diffeomorphism by construction. By lemma 5.15, we have that $(M_1' \circ M_0', \mathcal{A}'_\phi)$ is diffeomorphic to $(M_1' \circ M_0', \mathcal{A}')$, so in total we have the following diagram:



which shows that the two gluings are equivalent. Hence, the resulting composition is independent of choice of representative. □

5.3.4 Associativity of the composition

Given three cobordisms

$$M_0 : \Sigma_0 \Rightarrow \Sigma_1,\ M_1 : \Sigma_1 \Rightarrow \Sigma_2 \text{ and } M_2 : \Sigma_2 \Rightarrow \Sigma_3,$$

we would like to have the associativity property. That is,

$$(M_2 \circ M_1) \circ M_0 = M_2 \circ (M_1 \circ M_0).$$

Since we can pick disjoint collars near the boundary components, the different gluings do not "interact" with each other. Thus, the order we glue in does not matter which is to say that the gluing is associative.

5.3.5 The identity cobordism

We will now see that the cylinder acts as identity under gluing. Given a cobordism M, we can by corollary 5.14 decompose it as $M = M_{[\epsilon,1]} \circ M_{[0,\epsilon]}$, where $M_{[0,\epsilon]}$ is diffeomorphic to a cylinder over the in-boundary. Composing with a cylinder C over the in-boundary, we have that

$$M \circ C = (M_{[\epsilon,1]} \circ M_{[0,\epsilon]}) \circ C = M_{[\epsilon,1]} \circ (M_{[0,\epsilon]} \circ C) = M_{[\epsilon,1]} \circ M_{[0,\epsilon]} = M.$$

Here, we used lemma 5.16 which says that the composition of two cylinders are again a cylinder, and the associativity of the gluing. The same argument applies for composition of the left side. (Graphically, that is on the right side.)

5.4 The category $\mathbf{nCob^{or}}$

We are now ready to assemble everything into a category.

Definition 5.19. The (symmetric monoidal) *category of n-dimensional oriented cobordisms*, denoted $(\mathbf{nCob^{or}}, \amalg, \emptyset_{n-1}, \mathscr{T})$, is defined as follows:

1. **The objects** are closed and oriented $(n-1)$-dimensional manifolds.
2. **The morphisms** are n-dimensional oriented cobordism classes.
3. **Composition of morphisms** is defined as in definition 5.17.
4. **The identity morphism** on an object Σ is the class represented by the cylinder $\Sigma \times [0, 1]$.
5. **The monoidal product** is the disjoint union denoted $\amalg$.
6. **The neutral object** is the $(n-1)$-dimensional empty manifold $\emptyset_{n-1}$.
7. **The twist map** is the cobordism class represented by the twist cobordism $\mathscr{T}_{-,-}$

Note that we will often just write $\mathbf{nCob^{or}}$ when we mean $(\mathbf{nCob^{or}}, \amalg, \emptyset_{n-1}, \mathscr{T})$.

5.5 Topological Quantum Field Theories

Recall from the section on symmetric monoidal categories that $\mathbf{Vect}_{\Bbbk}$ is a symmetric monoidal category with the usual tensor product over $\Bbbk$ and the natural twist isomorphism $\sigma_{X,Y} : X \otimes Y \to Y \otimes X$ given by $x \otimes y \mapsto y \otimes x$.

Definition 5.20. An *(oriented) n-dimensional topological quantum field theory (TQFT)* is a symmetric monoidal functor

$$F : (\mathbf{nCob^{or}}, \amalg, \emptyset, \mathscr{T}) \to (\mathbf{Vect}_{\Bbbk}, \otimes, \Bbbk, \sigma)$$

We denote the functor category of such symmetric monoidal functors by

$$\mathbf{nTQFT}^{\mathbf{or}}_{\Bbbk} := \mathbf{Fun}^{\otimes}(\mathbf{nCob^{or}}, \mathbf{Vect}_{\Bbbk})$$

where the arrows are monoidal natural transformations.

Definition 5.21. Given a symmetric monoidal category $(\mathscr{C}, \otimes, I, \mathscr{T})$, a symmetric monoidal functor to $\mathbf{Vect}_{\Bbbk}$ is called a *linear representation of* $\mathscr{C}$. The category of all linear representations of $\mathscr{C}$ we denote by $\mathbf{Repr}_{\Bbbk}(\mathscr{C})$.

In other words, we can say that $\mathbf{nTQFT}^{\mathbf{or}}_{\Bbbk}$ is the category of linear representations of $\mathbf{nCob}^{\mathbf{or}}$.

Remark 5.22. By definition of being a symmetric monoidal functor, if $\mathscr{Z}$ is a TQFT, then it must map $\varnothing_n \mapsto \Bbbk$ and $\mathscr{T}_{X,Y} \mapsto \sigma_{\mathscr{Z}X,\mathscr{Z}Y}$.

Example 5.23. The trivial n-dimensional TQFT $\mathscr{Z}_0$ can be defined by sending every object to the ground field $\Bbbk$ and every morphism to the identity $\mathrm{id}_{\Bbbk}$.

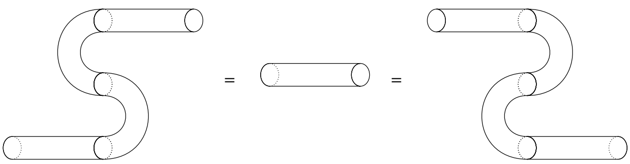
We will look at more interesting examples of TQFTs after we have proved the main theorem of this text.

5.5.1 Snake decomposition of the cylinder

We shall now prove a property of 2-dimensional TQFTs. Namely, that the image of the circle under a 2-dimensional TQFT is a finite dimensional vector space. Let $\mathscr{Z}$ be a 2-dimensional TQFT, and let A be the image of the circle under $\mathscr{Z}$.

Lemma 5.24. The vector space $A := \mathscr{Z}(S^1)$ is finite dimensional.

Proof. In $\mathbf{2Cob}^{\mathbf{or}}$ we have the following decompositions of the cylinder:



Let $\beta : A \otimes A^* \to \Bbbk$ be the image of the macaroni cobordism with two in-boundaries and let $\gamma : \Bbbk \to A^* \otimes A$ be the image of the macaroni cobordism with two out-boundaries. By functoriality of $\mathscr{Z}$ and by the decomposition in $\mathbf{2Cob}^{\mathbf{or}}$ we have in $\mathbf{Vect}_{\Bbbk}$ that

$$(\beta \otimes \mathrm{id}_A)\circ(\mathrm{id}_A \otimes \gamma) = \mathrm{id}_A = (\mathrm{id}_A \otimes \beta)\circ(\gamma \otimes \mathrm{id}_A).$$

The equality on the left translates into the following commutative diagram:

$$A \cong A \otimes \Bbbk \xrightarrow{\mathrm{id}_A \otimes \gamma} A \otimes A^* \otimes A \xrightarrow{\beta \otimes \mathrm{id}_A} \Bbbk \otimes A \cong A, \qquad A \cong A \otimes \Bbbk \xrightarrow{\mathrm{id}_A} \Bbbk \otimes A \cong A$$

Recall from definition 3.6 that is exactly what it means for β to be non-degenerate in A. Thus, by lemma 3.8, A is finite dimensional. □

Remark 5.25. Note that in the proof above, we identified $\mathcal{Z}\left(\overline{\Sigma}\right)$ with A^*. We do this using the map

$$\mathcal{Z}\left(\overline{\Sigma}\right) \xrightarrow{\cong} V^*$$
$$u \mapsto [\beta(-, u) : V \to \Bbbk],$$

which is an isomorphism by lemma 3.9 since β is non-degenerate.

Before moving to the next section, we remind ourselves of our main theorem: *the category of 2-dimensional TQFTs is equivalent to the category of commutative Frobenius algebra.* To prove this we will first need to understand $\mathbf{2Cob^{or}}$ a little better. This is exactly what we are going to do next.

5.6 Normal form and generators of 2Cob$^{\text{or}}$

Since we know the classification theorem for orientable 2-manifolds we will now use this to classify the morphisms in $\mathbf{2Cob^{or}}$. First, we shall consider the connected case and then move to cobordisms with potentially multiple connected components. What we obtain in the end is a normal form where we realize that we can build every 2-dimensional cobordism from six elementary building blocks using disjoint union and composition and conversely: given a two dimensional cobordism we can decompose into generators and write it in normal form.

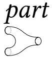
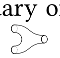
To make things simpler, we will instead of looking at the entire category $\mathbf{2Cob^{or}}$, just consider a skeleton of this category. Since we know that every closed and oriented 1-manifold is diffeomorphic to a disjoint union of circles (See 2.25.), we have a skeleton of $\mathbf{2Cob^{or}}$ with objects being such disjoint unions.

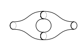
Definition 5.26. Let $\mathbf{Sk}(\mathbf{2Cob^{or}})$ denote the skeleton of $\mathbf{2Cob^{or}}$ with objects being disjoint unions of circles. For simpler notation, let $\mathbb{n}$ denote the disjoint union of n copies of the circle. We regard $\mathbb{0}$ as the empty 1-manifold $\varnothing_1$.

Note that in [Koc04] the same notation $\mathbf{2Cob^{or}}$ is used for the skeleton as well as the full category.

5.6.1 Normal form

Let us first consider connected cobordisms. We wish to build a cobordism given the number of incoming boundary components, the genus and the number of outgoing boundary components. We do this by the following procedure:

1. First, we use  to merge the incoming boundaries and call this the *incoming part* of the normal form. We do this by gluing the outgoing boundary of one  to the lower incoming boundary component of the following , and glue in the appropriate number of cylinders. In this way, the cylinders in the incoming part always comes on top of the pair of pants.

2. Then, we construct the given genus by composing handles . We call this part *the topological part* of the normal form.

3. Finally, we use  to split until we get the given number of outgoing boundary components and call this the *outgoing part* of the normal form. Again, we do this in a way such that cylinders always come on top of the reverse pair of pants.

Consider the following example with 3 incoming boundary components, genus 2 and 5 outgoing boundary components:

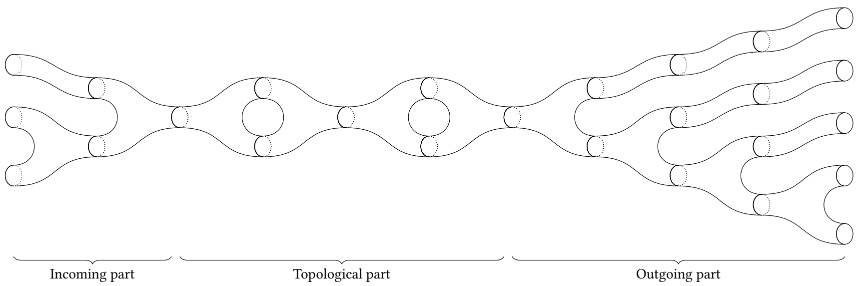


Figure 5.6.1: A cobordism in normal form.

If the incoming or outgoing part of the boundary is empty we just use  or , respectively. We add an additional example for the sake of completeness:

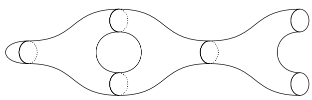
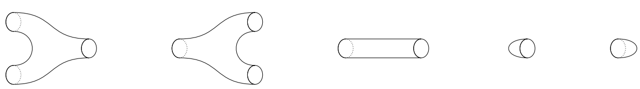


Figure 5.6.2: A cobordism in normal form with empty incoming boundary, genus 1 and 2 outgoing components.

5.6.2 Generators of $\mathbf{2Cob^{or}}$

By corollary 2.30, we know that two-dimensional cobordism are determined exactly by these three pieces of information and hence we have the following lemma:

Lemma 5.27. [Koc04, p. 65] Every connected 2-dimensional oriented cobordism is equivalent to one which is built out of the following cobordisms:

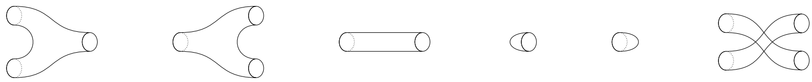


by the above construction.

For the case where we do not necessarily have a connected cobordism, we need to add one more generator, namely, the twist cobordism. The proof of the next lemma is presented very nicely in [Koc04]. We sketch the basic idea behind the proof: by a *permutation cobordism* we mean a disjoint union of twist cobordisms and cylinders (identity). We then show that every cobordism is equivalent to a composition

$P_0 \circ M \circ P_1$, where P_0 and P_1 are permutation cobordisms and M is the disjoint union of connected cobordisms (so that lemma 5.27 applies). An alternative approach using Morse theory is also given [Koc04, p. 68].

Lemma 5.28. [Koc04, p. 62] Every 2-dimensional cobordism is equivalent to one which is built by composition and disjoint union of the following generators:



In fact, given a 2-dimensional oriented cobordism M, then M is equivalent to a composition $P_0 \circ N \circ P_1$ where P_0 and P_1 are permutation cobordisms and N is the disjoint union of connected cobordisms in normal form.

Remark 5.29. Note that the decomposition $M = P_0 \circ N \circ P_1$ in lemma 5.28 is not unique. However, given two such normal forms of M, they differ only up permutations. See [Koc04, pp. 72-77] for more details on this.

6 Frobenius algebras and TQFTs

6.1 Relations in $\mathbf{2Cob^{or}}$

We will now employ our understanding of $\mathbf{2Cob^{or}}$ to establish some important relations there and observe what the corresponding relations in $\mathbf{Vect}_{\Bbbk}$ are. By corresponding, we mean the following: since a TQFT preserves the symmetric monoidal structure, the relations we have in $\mathbf{2Cob^{or}}$ translates into relations in $\mathbf{Vect}_{\Bbbk}$. This is exactly how we shall prove the following theorem, which is the first half of our main theorem (7.1):

Theorem 6.1. Let A be the image of the circle under a 2-dimensional TQFT. Then A is a commutative Frobenius algebra.

The proof of this theorem is the content of the next subsections. For the rest of this section, let $\mathscr{Z}$ be a 2-dimensional TQFT and let A be the image of the circle under $\mathscr{Z}$. The following table shows the generators of $\mathbf{2Cob^{or}}$ and their images under $\mathscr{Z}$ in $\mathbf{Vect}_{\Bbbk}$.

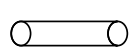
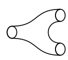
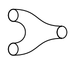
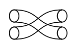
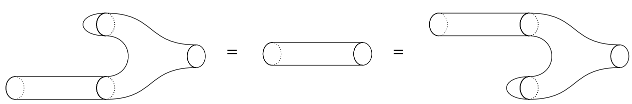
	Cylinder	Copair of pants	Pair of pants	Cap	Cup	Twist
$\mathbf{2Cob^{or}}$						
$\downarrow \mathscr{Z} \downarrow$						
$\mathbf{Vect}_{\Bbbk}$	Identity id_A	Comultiplication δ	Multiplication μ	Counit ϵ	Unit η	Twist $\mathscr{T}_{A,B}$
	$A \to A$	$A \to A \otimes A$	$A \otimes A \to A$	$A \to \Bbbk$	$\Bbbk \to A$	$A \otimes B \xrightarrow{\cong} B \otimes A$

Figure 6.1.1: The generators of $\mathbf{2Cob^{or}}$ and their corresponding maps in $\mathbf{Vect}_{\Bbbk}$.

We will use the suggestive names of the maps in $\mathbf{Vect}_{\Bbbk}$ from this table throughout this section. Note that we will often omit identity cylinders in our drawings from now on. Also, when we write "=" between drawings of cobordisms, we mean that they represents the same class. (I.e., they are equivalent as cobordisms.)

6.1.1 Unit relation

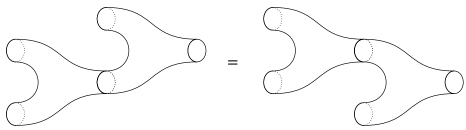


This relation implies the following relation on the image under $\mathscr{Z}$:

$$\mu\circ(\mathrm{id}_A \otimes \eta) = \mathrm{id}_A = \mu\circ(\eta \otimes \mathrm{id}_A),$$

which on elements just means that $1_A x = x = x 1_A$. This is exactly the condition that the two triangles in the commutative diagram defining the axioms for a $\Bbbk$-algebra commutes. (See definition 3.1).

6.1.2 Associativity relation

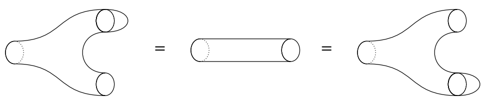


In $\mathbf{Vect}_\Bbbk$ this corresponds to the relation

$$\mu\circ(\mu \otimes \mathrm{id}_A) = \mu\circ(\mathrm{id}_A \otimes \mu)$$

or simply $(xy)z = x(yz)$. This exactly says that the diamond the defining diagram of a $\Bbbk$-algebra commutes. (See definition 3.1.)

6.1.3 Counit relation

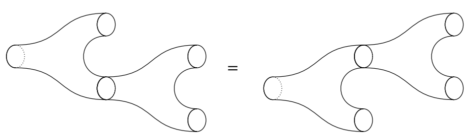


In $\mathbf{Vect}_\Bbbk$ this corresponds to the relation

$$(\mathrm{id}_A \otimes \epsilon)\circ\delta = \mathrm{id}_A = (\epsilon \otimes \mathrm{id}_A)\circ\delta$$

Which is exactly the two triangles in the commutative diagram defining a coalgebra over $\Bbbk$. (See definition 3.13.)

6.1.4 Coassociativity relation



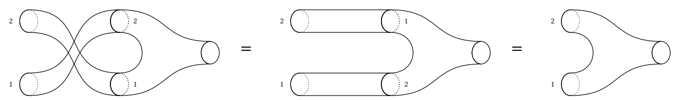
In $\mathbf{Vect}_\Bbbk$ this corresponds to the relation

$$(\delta \otimes \mathrm{id}_A)\circ\delta = (\mathrm{id}_A \otimes \delta)\circ\delta$$

Giving us the coassociativity of A. (See definition 3.1.)

6.1.5 Commutativity relation

Since TQFTs are symmetric monoidal functors, they must respect the symmetric structure. That is, the twist cobordism is sent to the natural twist isomorphism in $\mathbf{Vect}_{\mathbb{k}}$. We have the following relation in $\mathbf{2Cob}^{\mathbf{or}}$:



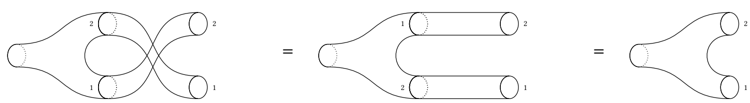
This induces the following commutative diagram in $\mathbf{Vect}_{\mathbb{k}}$:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\ \mu\ } & A \\ {\scriptstyle \sigma_{A,A}} \big\downarrow & \nearrow_{\mu} & \\ A \otimes A & & \end{array}$$

Which is to say that μ is commutative. (See definition 3.3.)

6.1.6 Cocommutativity relation

We also have that the comultiplication is commutative:



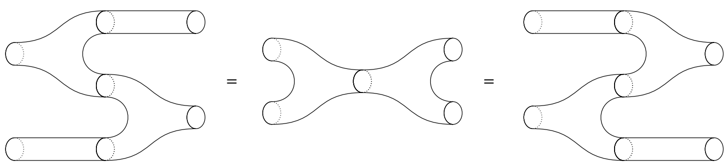
because this translates into the following relation:

$$\begin{array}{ccc} A & \xrightarrow{\ \delta\ } & A \otimes A \\ & {}_{\delta}\searrow & \big\downarrow {\scriptstyle \sigma_{A,A}} \\ & & A \otimes A \end{array}$$

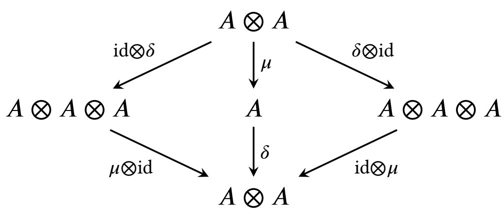
Which is to say that δ is cocommutative. (See definition 3.14.)

6.1.7 The Frobenius relation

The following relation holds in $\mathbf{2Cob}^{\mathbf{or}}$:



This relation translates into the diagram



in **Vect**$_{\mathbb{k}}$, which is exactly the Frobenius relation. (See definition 3.24.) Thus, by the previous relations together with lemma 3.28 we have that A is a Frobenius algebra. Hence, we have proved theorem 6.1. □

7 Proof of the main theorem

We have now arrived at our main goal of this text, which is to establish the following theorem:

Theorem 7.1. Let $\mathscr{Z}$ and $\mathscr{Q}$ be two 2-dimensional TQFTs and $\alpha : \mathscr{Z} \Rightarrow \mathscr{Q}$ be a monoidal natural transformation. The functor $F : \mathbf{Sk2TQFT}^{\mathbf{or}}_{\Bbbk} \to \mathbf{cFA}_{\Bbbk}$ defined by

$$\begin{aligned} \mathscr{Z} &\mapsto \mathscr{Z}(\mathbb{1}) \\ \alpha &\mapsto [\alpha_{\mathbb{1}} : \mathscr{Z}(\mathbb{1}) \to \mathscr{Q}(\mathbb{1})], \end{aligned}$$

is an isomorphism of categories.

Here, $\mathbf{Sk2TQFT}^{\mathbf{or}}_{\Bbbk}$ denotes the category of TQFTs from the skeleton of $\mathbf{2Cob}^{\mathbf{or}}$ into $\mathbf{Vect}_{\Bbbk}$. We will use this notation throughout this section.

When we have proved theorem 7.1, we get the following corollary for free:

Corollary 7.2. There is an equivalence of categories

$$\mathbf{2TQFT}^{\mathbf{or}}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk}.$$

Let us assume theorem 7.1 and prove this corollary.

Proof. From theorem 7.1, we have that $\mathbf{Sk2TQFT}^{\mathbf{or}}_{\Bbbk} \cong \mathbf{cFA}_{\Bbbk}$. Since $\mathbf{Sk}(\mathbf{2Cob}^{\mathbf{or}})$ is equivalent to $\mathbf{2Cob}^{\mathbf{or}}$ (by lemma 4.24), we have by lemma 4.25, that $\mathbf{Sk2TQFT}^{\mathbf{or}}_{\Bbbk} \simeq \mathbf{2TQFT}^{\mathbf{or}}_{\Bbbk}$. Since equivalence of categories is transitive (by lemma 4.20), we have that

$$\mathbf{2TQFT}^{\mathbf{or}}_{\Bbbk} \simeq \mathbf{cFA}_{\Bbbk}.$$

□

The proof of theorem 7.1 is the content of the rest of this section, and we shall break it up into multiple lemmas.

Lemma 7.3. F is well-defined.

Proof. From theorem 6.1, we know that $\mathscr{Z}(\mathbb{1})$ is a Frobenius algebra. Now, we have to make sure that $\alpha_{\mathbb{1}}$ actually is a morphism of Frobenius algebras. This will rely on the axioms for monoidal natural transformations. Note in particular that one of the axioms for monoidal natural transformations is that $\alpha_{\mathbb{0}} : \Bbbk \to \Bbbk$ is the identity on $\Bbbk$. To simplify notation a bit let $A := \mathscr{Z}(\mathbb{1})$ and $B := \mathscr{Q}(\mathbb{1})$. Also, since the objects in $\mathbf{Sk}(\mathbf{2Cob}^{\mathbf{or}})$ are disjoint unions $\mathbb{n} = \coprod_{i=0}^{n} S^1$ and TQFTs are monoidal, we have that $\alpha_{\mathbb{n}} = \alpha_{\mathbb{1}}^{\otimes n} = \alpha_{\mathbb{1}} \otimes \ldots \otimes \alpha_{\mathbb{1}}$ which means that every component of α is determined by $\alpha_{\mathbb{1}}$ by taking the n-fold tensor product. So by α being monoidal we have that the following cluster of diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\alpha_{\mathbb{1}} \otimes \alpha_{\mathbb{1}}} & B \otimes B \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ A & \xrightarrow[\alpha_{\mathbb{1}}]{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\alpha_{\mathbb{1}}} & B \\ {\scriptstyle \eta_A}\uparrow & & \uparrow{\scriptstyle \eta_B} \\ \Bbbk & \xrightarrow[\alpha_{\mathbb{0}} = \mathrm{id}_{\Bbbk}]{} & \Bbbk \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathbb{1}}} & B \\ {\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \delta_B} \\ A \otimes A & \xrightarrow[\alpha_{\mathbb{1}} \otimes \alpha_{\mathbb{1}}]{} & B \otimes B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\alpha_{\mathbb{1}}} & B \\ {\scriptstyle \epsilon_A}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\ \Bbbk & \xrightarrow[\alpha_{\mathbb{0}} = \mathrm{id}_{\Bbbk}]{} & \Bbbk \end{array}$$

Here we use the notation from table 6.1.1, i.e. μ_A is the image of  under $\mathscr{Z}$, δ_A the image of  and so on. In the same way we get the algebraic structure of B as the images of these generators under $\mathscr{Q}$. That the diagrams above commute is exactly what it means for $\alpha_{\mathbb{1}}$ to be a morphism of Frobenius algebras. □

The idea for the other direction is as follows: send a Frobenius algebra to the TQFT that sends the circle to this algebra.

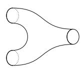
Construction. We define the functor

$$G\colon \mathbf{cFA}_{\Bbbk} \to \mathbf{Sk2TQFT}_{\Bbbk}^{\mathbf{or}}$$

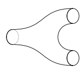
on objects as follows: a commutative Frobenius algebra A is sent to the symmetric monoidal functor $\mathscr{Z}_A\colon \mathbf{Sk}(\mathbf{2Cob^{or}}) \to \mathbf{Vect}_{\Bbbk}$ induced by

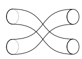
$\mathbb{n} \mapsto A^{\otimes n}$

 $\mapsto [\eta_A\colon \Bbbk \to A]$

 $\mapsto [\mu_A\colon A \otimes A \to A]$

 $\mapsto [\epsilon_A\colon A \to \Bbbk]$

 $\mapsto [\delta_A\colon A \to A \otimes A]$

 $\mapsto [\sigma_{A,A}\colon A \otimes A \to A \otimes A]$

 $\mapsto [\mathrm{id}_A\colon A \to A]$.

That this assignment extends to $\mathbf{Sk}(\mathbf{2Cob^{or}})$ is the content of the next lemma.

Lemma 7.4. Let A be a commutative Frobenius algebra. Then the functor $\mathscr{Z}_A := G(A)$ is fully and uniquely determined by the above assignment.

Proof. If given an arbitrary 2-dimensional cobordism $M \in \mathbf{Sk}(\mathbf{2Cob^{or}})$, we define $\mathscr{Z}_A$ as follows:

$$\mathscr{Z}_A(M) := \mathscr{Z}_A(P_0) \circ \mathscr{Z}_A(N) \circ \mathscr{Z}_A(P_1),$$

where $P_0 \circ N \circ P_1$ is M in the normal form described in lemma 5.28. The permutation cobordisms[16] P_0 and P_1 are not unique. Also, N, the disjoint union of connected cobordisms in normal form, is only unique up to the ordering of factors in the disjoint union. Thus, we have to check that if given any two normal forms

$$P_0 \circ N \circ P_1 \quad \text{and} \quad T_0 \circ N' \circ T_1,$$

[16] Recall that a permutation cobordism is the disjoint union and composition of identities and twist cobordisms.

of M, then their images under $\mathcal{Z}_A$ are the same. We divide this proof into the following three cases:

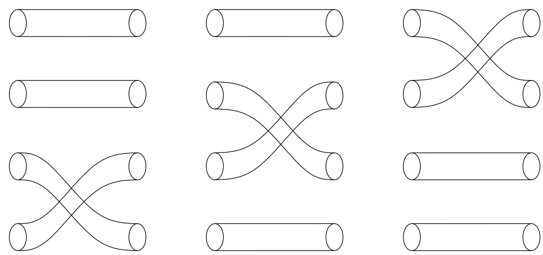
(i) M is a permutation cobordism,

(ii) M is connected or a disjoint union of connected cobordisms (see definition 5.10) in normal form,

(iii) M is an arbitrary 2-dimensional cobordism

and show that $\mathcal{Z}_A(M)$ is well-defined in each of these cases.

Case (i): Suppose $M : \mathbb{n} \Rightarrow \mathbb{n}$ is a permutation cobordism. Observe that we can achieve any permutation by composing adjacent transposition cobordisms (twist cobordisms interchanging two adjacent circles). To make this more clear, let us draw the pictures of all three elementary transposition cobordisms when $n = 4$:



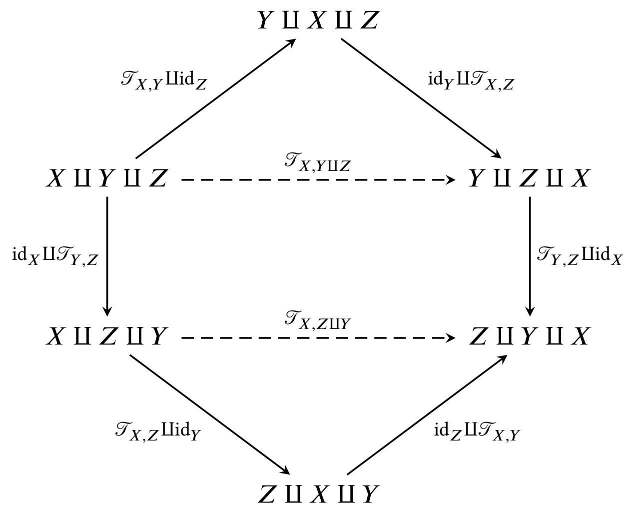
Let $P^{i,i+1}$ denote the elementary transposition that interchanges the ith and $(i+1)$th circle. (I.e., $P^{2,3}$ is the middle one in the above picture.) We define $\mathcal{Z}_A$ on these elementary transpositions as follows:

$$\mathcal{Z}_A : \; P^{i,i+1} \mapsto \mathrm{id}^{\otimes(i-1)} \otimes \sigma_{A,A} \otimes \mathrm{id}^{\otimes(n-i-1)}$$

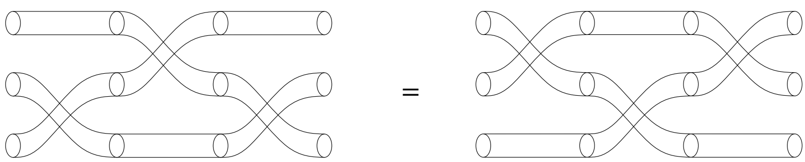
Let us write M as a composition of such adjacent transpositions:

$$M = P_k \circ P_{k-1} \circ \ldots \circ P_0.$$

We then define $\mathcal{Z}_A(M)$ to be $\mathcal{Z}_A(P_k) \circ \mathcal{Z}_A(P_{k-1}) \circ \ldots \circ \mathcal{Z}_A(P_0)$. Next, we are going to show that this is indeed well-defined. Consider the following diagram:

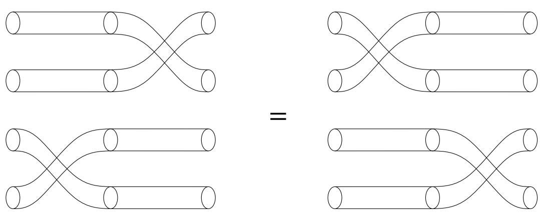


The center square is obtained by applying the first diagram in the definition of a symmetric monoidal category (See 4.11.) on the morphisms id_X and $\mathscr{T}_{Y,Z}$. Then we used the third diagram in definition 4.11 to obtain the upper and lower triangles. Thus, the above diagram commutes. We draw the solid arrows as a relation in $\mathbf{2Cob^{or}}$:



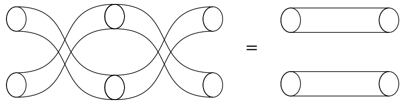
(1)

It is clear that the following relation also holds in $\mathbf{2Cob^{or}}$:



(2)

Finally, from the second diagram in definition 4.11, we have the relation:



(3)

Now, these are the generators and defining relations of the symmetric group on n letters, denoted $\mathfrak{S}_n$. Hence, we can identify the subcategory in $\mathbf{Sk}(\mathbf{2Cob^{or}})$ consisting of the single object $\mathbb{n}$ together with all permutation cobordisms $\mathbb{n} \Rightarrow \mathbb{n}$, with $\mathfrak{S}_n$. Since we have defined $\mathcal{Z}_A$ on the generators $P^{i,i+1}$, we have that $\mathcal{Z}_A(M)$ respects relations (1)-(3) by construction since the twist cobordism is sent to the twist isomorphism $\sigma_{A,A}$. That is, it does not depend on *how* we write M as a composition of elementary transpositions.

Case (ii): Let M be a connected cobordism. Then, by lemma 5.27, it can be uniquely reduced to normal form (containing no twist cobordisms). Say

$$M = M_k \circ M_{k-1} \circ \ldots \circ M_0$$

and

$$M_j = \coprod_{i=0}^{q_j} M_j^i$$

for $i = 0, 1, \ldots, k$. Then, we define the image of M_j under $\mathcal{Z}_A$ to be

$$\bigotimes_{i=0}^{q_j} \mathcal{Z}_A(M_j^i)$$

and the image of M to be

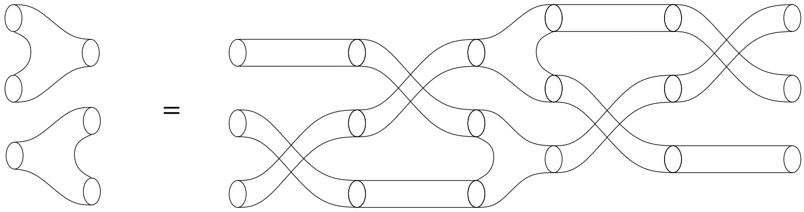
$$\mathcal{Z}_A(M_k) \circ \mathcal{Z}_A(M_{k-1}) \circ \ldots \circ \mathcal{Z}_A(M_0).$$

If M is the disjoint union of connected cobordisms in normal form, say $M = M' \amalg M''$, then we define $\mathcal{Z}_A(M) = \mathcal{Z}_A(M') \otimes \mathcal{Z}_A(M'')$.

Case (iii): Suppose $M : \mathbb{n} \Rightarrow \mathbb{m}$ is an arbitrary 2-dimensional cobordism in $\mathbf{Sk}(\mathbf{2Cob^{or}})$ and pick two decompositions

$$P_0 \circ N \circ P_1 \quad \text{and} \quad T_0' \circ N' \circ T_1'$$

of M. We know that N and N' only differs by the ordering of their components. We start by reordering the disjoint union N' to make the order the same as in N. We do this by composing N with permutations T_0'' and T_1'' on both sides so that we obtain $N' = T_0'' \circ N \circ T_1''$. To make this more clear, consider the following picture where we switch the order of the two components:



Let $T_0 := T_0' \circ T_0''$ and $T_1 := T_1'' \circ T_1'$. In total, we now have that

$$M = P_0 \circ N \circ P_1 = T_0 \circ N \circ T_1.$$

We will now argue that P_i and T_i are equivalent as cobordisms for $i = 0, 1$: since N is the disjoint union of cobordisms in normal form, by lemma 5.27 there is no twist cobordisms connected to only a single connected component (because connected components can be made without using the twist cobordism). Thus, if P_0 connects some Σ in the in-boundary of M to Σ' in the in-boundary of N, then T_0 also connects Σ to Σ'. Hence they are the same permutations (modulo the relation (1)-(3) from case (i)). It then follows by case (i) that $\mathscr{Z}_A(P_0) = \mathscr{Z}_A(T_0)$. The same argument apply for P_1 and T_1. Thus, $\mathscr{Z}_A$ extends to $\mathbf{Sk}(\mathbf{2Cob^{or}})$. This completes our proof. □

On morphisms we define G as follows: let $f : A \to B$ be a morphism of Frobenius algebras and define the image under G like this:

$$G(f)_{\mathbb{n}} := \alpha_{\mathbb{n}} := f^{\otimes n} : G(A)(\mathbb{n}) \to G(B)(\mathbb{n}).$$ [17]

For simplicity, denote this family of maps $\alpha = \{\alpha_{\mathbb{n}}\}_{\mathbb{n} \in \mathbf{Sk}(\mathbf{2Cob^{or}})}$.

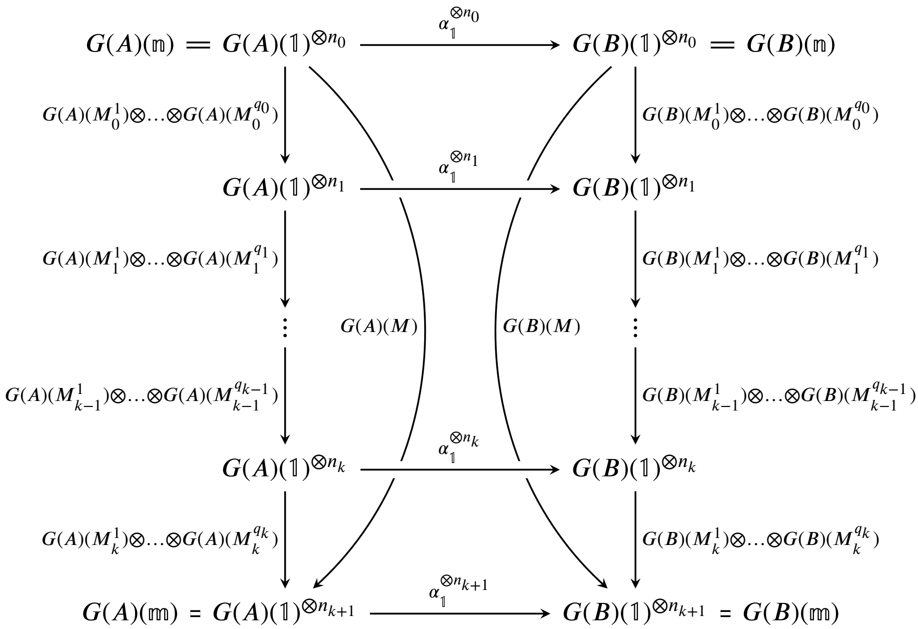
Lemma 7.5. *G is well-defined.*

Proof. We just proved that the image of A is a well-defined TQFT. What remains to check is that α is natural. Let $M : \mathbb{n} \Rightarrow \mathbb{m}$ be the composition $M = M_k \circ M_{k-1} \circ \dots M_0$ where each M_j is a disjoint union of q_j generators in $\mathbf{2Cob^{or}}$. That is,

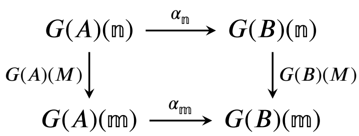
$$M_j = \coprod_{i=1}^{q_j} M_j^i.$$

For $j = 0, 1, \dots, k$, let n_j be the number of incoming boundary components of M_j, and let n_{k+1} be the number of outgoing outgoing boundary components of M_k. Note that $n = n_0$ and $m = n_{k+1}$. Since f is a morphism of Frobenius algebras, it is compatible with the images of the generators of $\mathbf{2Cob^{or}}$. The compability with the twist map follows by the naturality required of $\sigma_{A,A}$. Therefore we have that every square in the following diagram commutes:

[17] By $f^{\otimes n}$ we mean the n-fold tensor product of f. That is, $f \otimes f \otimes \dots \otimes f$, n times.



And hence, the following diagram:



commutes as well. This shows that α is indeed natural. $\square$

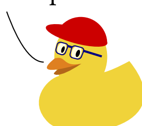
Note that we used the fact that $G(A)$ (and $G(B)$) is monoidal to obtain the diagram in the proof above. In particular, we used the following two equalities:

(i) $G(A)(M_j) = G(A)\left(\coprod_{i=1}^{q_j} M_j^i\right) = \bigotimes_{i=1}^{q_j} G(A)(M_j^i)$

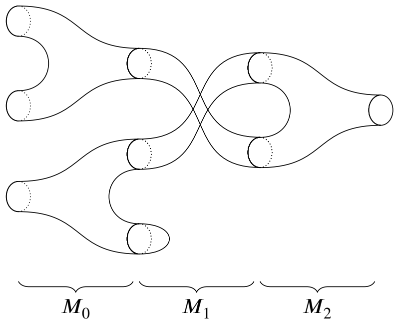
(ii) $G(A)(\mathbb{n}) = G(A)\left(\coprod_{i=1}^{n} \mathbb{1}\right) = \bigotimes_{i=1}^{n} G(A)(\mathbb{1}) = G(A)(\mathbb{1})^{\otimes n}$.

We also used that α is a monoidal natural transformation. This is by definition since $\alpha_{\mathbb{n}} = f^{\otimes n} = \alpha_{\mathbb{1}}^{\otimes n}$.

Now here's an example to better explain what the duck is going on!



Example 7.6. Consider the following composite cobordism $M = M_2 \circ M_1 \circ M_0 : \mathbb{3} \Rightarrow \mathbb{1}$:



j	**0**	**1**	**2**	**3**
$\mathbf{n_j}$	3	3	2	1
$\mathbf{q_j}$	2	2	1	-

The images under $G(A)$ and $G(B)$ translates into the following diagram:

$$\begin{array}{ccccccc}
G(A)(\mathbb{3}) & = & A \otimes A \otimes A & \xrightarrow{f \otimes f \otimes f} & B \otimes B \otimes B & = & G(B)(\mathbb{3}) \\
 & & \downarrow{\scriptstyle \delta_A \otimes \mu_A} & & \downarrow{\scriptstyle \delta_B \otimes \mu_B} & & \\
G(A)(\mathbb{3}) & = & A \otimes A \otimes A & \xrightarrow{f \otimes f \otimes f} & B \otimes B \otimes B & = & G(B)(\mathbb{3}) \\
 & & \downarrow{\scriptstyle \epsilon_A \otimes \sigma_{A,A}} & & \downarrow{\scriptstyle \epsilon_B \otimes \sigma_{B,B}} & & \\
G(A)(\mathbb{2}) & = & A \otimes A & \xrightarrow{f \otimes f} & B \otimes B & = & G(B)(\mathbb{2}) \\
 & & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \mu_B} & & \\
G(A)(\mathbb{1}) & = & A & \xrightarrow{f} & B & = & G(B)(\mathbb{1})
\end{array}$$

in which every square commutes since f is a morphism of Frobenius algebras and by the naturality of the twist map. Observe that the vertical compositions on the left and right are the images of M under $G(A)$ and $G(B)$, respectively. So the outer diagram is the diagram expressing the naturality of α:

$$\begin{array}{ccc}
G(A)(\mathbb{3}) & \xrightarrow{\alpha_3} & G(B)(\mathbb{3}) \\
{\scriptstyle G(A)(M)}\downarrow & & \downarrow{\scriptstyle G(B)(M)} \\
G(A)(\mathbb{1}) & \xrightarrow{\alpha_{\mathbb{1}}} & G(B)(\mathbb{1})
\end{array}$$

We shall now see that F and G are in fact inverses of each another.

Lemma 7.7. $F \circ G = \mathrm{id}_{\mathbf{cFA}_{\Bbbk}}$ and $G \circ F = \mathrm{id}_{\mathbf{Sk2TQFT}^{\mathbf{or}}_{\Bbbk}}$.

Proof. Let A be a commutative Frobenius algebra over $\Bbbk$. We just plug in the definitions of F and G to see that the composition is in fact the identity functor:

$$F \circ G(A) = F\left(\left[\mathcal{Z}_A : \mathbb{1} \mapsto A\right]\right) = \mathcal{Z}_A(\mathbb{1}) = A.$$

Conversely, let $\mathcal{Z}$ be a TQFT. Since $\mathcal{Z}$ is monoidal it is completely determined on objects by where it sends the circle. Hence, the composition $G \circ F$ get us back to where we started:

$$G \circ F(\mathcal{Z}) = G(\mathcal{Z}(\mathbb{1})) = \left[\mathcal{Z} : \mathbb{1} \mapsto \mathcal{Z}(\mathbb{1})\right] = \mathcal{Z}.$$

Next we have to show that we have the identity on morphisms as well. So let $f : A \to B$ be a morphism of Frobenius algebras. We plug in the definition of G and F on morphisms and obtain the desired result:

$$F \circ G(f) = F\left(\left\{f^{\otimes n}\right\}_{n \in \mathbb{N}}\right) = f^{\otimes 1} = f$$

Conversely, let $\alpha = \left\{\alpha_{\mathbb{n}}\right\}$ be a monoidal natural transformation between two TQFTs. Then,

$$G \circ F(\alpha) = G(\alpha_{\mathbb{1}}) = \left\{\alpha_{\mathbb{1}}^{\otimes n}\right\}_{n \in \mathbb{N}} = \left\{\alpha_{\mathbb{n}}\right\}_{\mathbb{n} \in \mathbf{Sk}(\mathbf{2Cob}^{\mathbf{or}})}.$$

To get the last equality we used that α is monoidal. □

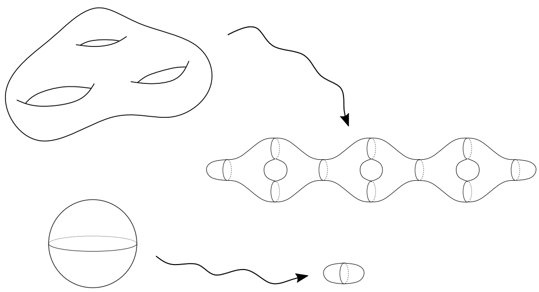
Now, that completes the proof of theorem 7.1! □

8 Invariants of manifolds

We will now have a look at some examples where we use two-dimensional TQFTs to produce invariants of manifolds. Since we now have proved theorem 7.1, we know that choosing a Frobenius algebra A completely determines a 2-dimensional TQFT. We denote this TQFT by $\mathscr{Z}_A$.

The idea is as follows: given a closed and oriented 2-manifold M, we can view it as a morphism in $\mathrm{Hom}_{\mathbf{2Cob}^{\mathbf{or}}}(\varnothing, \varnothing)$. Thus, we have a $\Bbbk$-linear map $\mathscr{Z}_A(M) \colon \Bbbk \to \Bbbk$ and we denote the image of $1_\Bbbk$ under this map by $\mathrm{inv}_A(M)$. If M is diffeomorphic to N we automatically have that $\mathscr{Z}_A(M) = \mathscr{Z}_A(N)$ since they represent the same cobordism class (there is no boundary to preserve).

To calculate the invariant, we cut M into pieces where each piece is a (possibly disjoint union of) generator(s) of $\mathbf{2Cob}^{\mathbf{or}}$.



Using table 6.1.1 we can calculate these maps to find $\mathrm{inv}_A(M)$.

Example 8.1. **(A trivial TQFT)** Let $A = \Bbbk$ be endowed with the Frobenius form $\epsilon := \mathrm{id}_\Bbbk \colon \Bbbk \to \Bbbk$. We construct the non-degenerate pairing $\beta := \epsilon \circ \mu = \mu$ and the corresponding co-pairing $\gamma \colon \Bbbk \to \Bbbk \otimes \Bbbk$ defined by $1 \mapsto 1 \otimes 1$. By calculation we then find that the comultiplication $\delta := (\mu \otimes \mathrm{id}_\Bbbk) \circ (\mathrm{id}_\Bbbk \otimes \gamma)$ is given by $1 \mapsto 1 \otimes 1$.

The handle operator $H := \mu \circ \delta$ is just the identity map on $\Bbbk$. Thus, $\mathrm{inv}_A(M) = 1$ so nothing interesting happens here.

Note that in the above example we could have choosen ϵ to be any non-zero $\Bbbk$-linear map because the only ideals in $\Bbbk$ are (0) and $\Bbbk$. This is what we are going to do in the next example.

Example 8.2. Let $A = \Bbbk$ be endowed with the Frobenius form $\epsilon \colon 1 \mapsto r$ for some non-zero $r \in \Bbbk$.

By setting $\beta := \epsilon \circ \mu$, one finds that the comultiplication δ is given by

$$a \mapsto r^{-1} a \otimes 1,$$

so that the handle operator $H := \mu \circ \delta$ is multiplication by r^{-1}. Let $N = \coprod_{j=1}^{n} N_j$ where each N_j is a surface of genus g_j. Let us calculate the map $\mathscr{Z}_A(N_j)$:

$$1_\Bbbk \overset{\eta}{\mapsto} 1 \overset{\delta}{\mapsto} r^{-1} \otimes 1 \overset{\mu}{\mapsto} r^{-1} \overset{\delta}{\mapsto} \dots \overset{\mu}{\mapsto} r^{-g_j} \overset{\epsilon}{\mapsto} r^{1-g_j}.$$

Hence, $\mathrm{inv}_A(N_j) = r^{1-g_j}$. It follows that

$$\mathcal{Z}_A(N)(1_{\Bbbk}) = r^{n-\sum_j g_j}.$$

In other words, $\mathrm{inv}_A(N)$ is the number of connected components of N minus the genus of N.

Example 8.3. (A hole counting TQFT) Let M be a closed, connected, and oriented 2-manifold of genus g, and let $\mathcal{Z}_A$ be the 2-dimensional TQFT that sends the circle to

$$A = \Bbbk\left[\left({}^{\mathbb{Z}}\!/\!_{3\mathbb{Z}}\right)^{\times}\right].$$

From example 3.21, we know that A is a Frobenius algebra and that $\mathcal{Z}_A(M)(1_{\Bbbk}) = 2^g$.

Thus, $\mathrm{inv}_A(M) = 2^g$ so we have recovered the genus.

Remark 8.4. If we let $\Bbbk$ be the finite field $\mathbb{F}_{31}$ then we count holes modulo 5. To see this, let M be a closed surface of genus $5m+k$ where $k = 0, \dots, 4$. Then we have that $\mathrm{inv}_A(M) = 2^{5m+k} = 32^m 2^k = 2^k$.

Example 8.5. Let $A = {}^{\Bbbk[t]}\!/\!_{(t^2-1)}$ be endowed with the Frobenius form

$$\begin{aligned}\epsilon :\ A &\to \Bbbk\\ a + bt &\mapsto b.\end{aligned}$$

The pairing $\beta := \epsilon\circ\mu$ is given by $(a+bt)\otimes(c+dt) \mapsto ad+bc$ and the copairing γ by $1_{\Bbbk} \mapsto t\otimes 1 + 1\otimes t$. As usual, we define the comultiplication in terms of γ and μ as $\delta := (\mu\otimes \mathrm{id}_{\Bbbk})\circ(\mathrm{id}_{\Bbbk}\otimes\gamma)$. Calculations gives that δ is defined by

$$a + bt \mapsto b\otimes 1 + at\otimes 1 + a\otimes t + bt\otimes t.$$

The handle operator $H := \mu\circ\delta$ is multiplication by $2t$ so that the composition of n handles, denoted H^n, is multiplication by $(2t)^n$. Now, let M be a connected surface of genus n. Then, we have that

$$1_{\Bbbk} \overset{\eta}{\mapsto} 1 \overset{H^n}{\mapsto} (2t)^n = \begin{cases} 2^n t & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases} \overset{\epsilon}{\mapsto} \begin{cases} 2^n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

That is, $\mathrm{inv}_A(M)$ counts only odd number of holes. If M is disconnected (i.e., a disjoint union of connected components), then $\mathrm{inv}_A(M) = 0$ if and only if some component is of even genus. If every component of M is of odd genus, then $\mathrm{inv}_A(M) = 2^g$ where g is the sum of genera.

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