Multicomplexes over a field

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Multicomplexes — Definition

A *multicomplex* $(M, D_\bullet)$ (over a field $\mathbb{K}$) consists of

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In addition, we require

$$D_0 D_n + D_1 D_{n-1} + \cdots + D_{n-1} D_1 + D_n D_0 = 0$$

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for all $n \geqslant 0$.

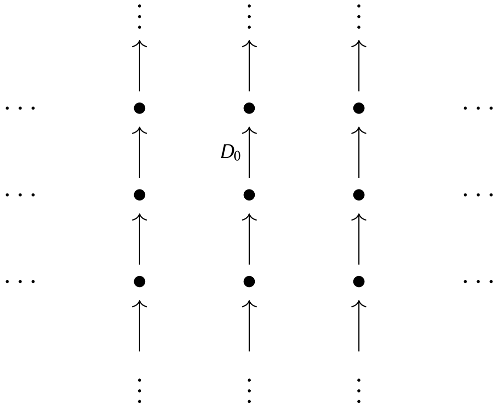
More compactly,

$$\boxed{\sum_{p+q=n} D_p D_q = 0.}$$

Visualising multicomplexes

$$
\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
\cdots & M^{p-1,q+1} & M^{p,q+1} & M^{p+1,q+1} & \cdots \\
\cdots & M^{p-1,q} & M^{p,q} & M^{p+1,q} & \cdots \\
\cdots & M^{p-1,q-1} & M^{p,q-1} & M^{p+1,q-1} & \cdots \\
 & \vdots & \vdots & \vdots &
\end{array}
$$

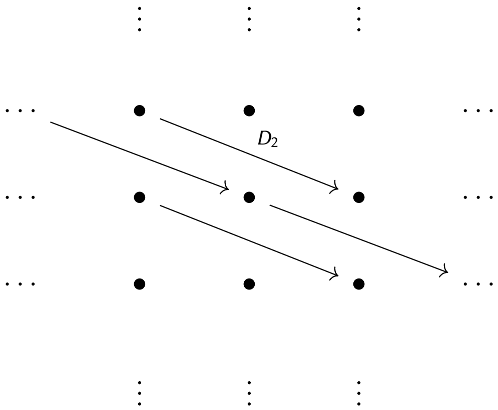
Visualising multicomplexes: D_0



Visualising multicomplexes: D_1

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
\cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \xrightarrow{D_1} & \bullet & \longrightarrow & \cdots \\
\cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Visualising multicomplexes: D_2



Defining relations

$$\boxed{\sum_{p+q=n} D_p D_q = 0.}$$

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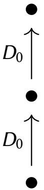
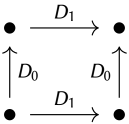


n=1

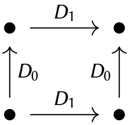
$$D_1 D_0 = -D_0 D_1$$

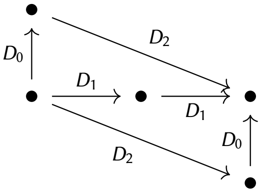


Defining relations

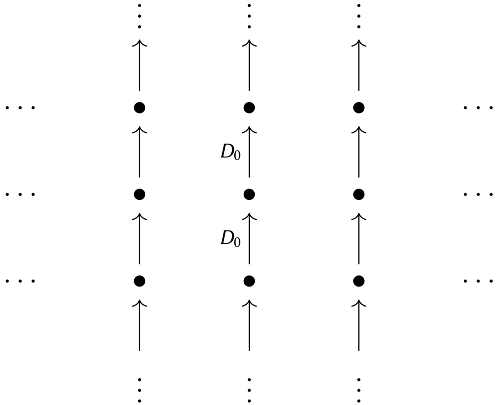
$$\boxed{\sum_{p+q=n} D_p D_q = 0.}$$

n=0	n=1	n=2
$D_0 D_0 = 0$	$D_1 D_0 = -D_0 D_1$	$D_1 D_1 = -(D_0 D_2 + D_2 D_0)$





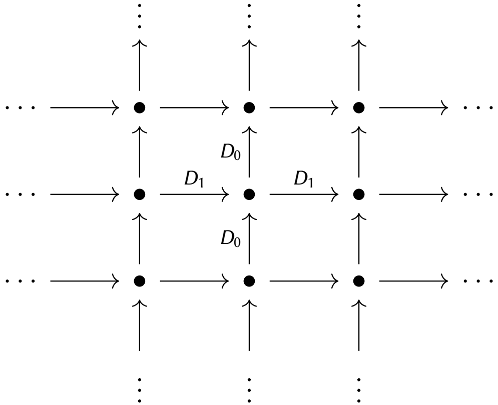
Example: Chain complex of (graded) vector spaces



$$D_1 = D_2 = \ldots = 0$$

$$\implies D_0 D_0 = 0$$

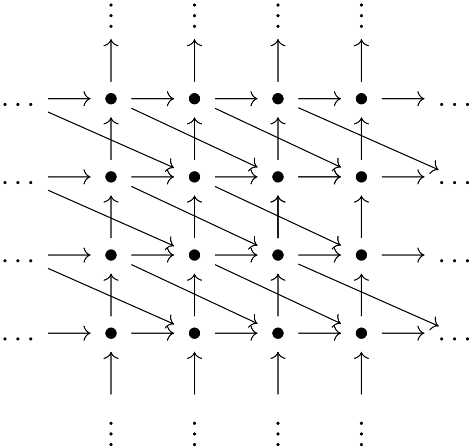
Example: Double complex



$$D_2 = D_3 = \ldots = 0$$

$$\Longrightarrow \quad D_0 D_0 = 0 \quad D_1 D_1 = 0 \quad D_0 D_1 + D_1 D_0 = 0$$

Example: "Homotopy double complex"



$$D_3 = D_4 = \ldots = 0$$

$$\Longrightarrow \quad D_0 D_0 = 0 \quad D_0 D_1 + D_1 D_0 = 0 \quad D_1 D_1 = -D_0 D_2 - D_2 D_0$$

$$D_1 D_2 + D_2 D_1 = 0 \quad D_2 D_2 = 0$$

Forgetting the higher structure

The underlying chain complex...

Forget the higher differentials: $(M, D_\bullet) \mapsto (M, D_0)$.

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The underlying chain complex...

Forget the higher differentials: $(M, D_\bullet) \mapsto (M, D_0)$.

...and its homology complex

The homology $H(M) = H(M, D_0)$ is defined degreewise:

$$H^{p,q}(M) = \frac{Z^{p,q}(M)}{B^{p,q}(M)} = \frac{\ker D_0 \colon M^{p,q} \to M^{p,q+1}}{\operatorname{im} D_0 \colon M^{p,q-1} \to M^{p,q}}$$

We equip $H(M)$ with the trivial differential.

Homotopy equivalence

Remark

Over a **field**, it is always true that $(M, D_0) \simeq (H(M), 0)$ as chain complexes.

Homotopy equivalence

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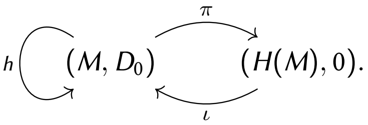
In other words, we can always find maps



with

$$\iota\pi - \mathrm{id}_M = D_0 h + h D_0 \quad \text{and } \pi\iota = \mathrm{id}_{H(M)} .$$

The Homotopy Transfer Theorem (HTT)

Given a homotopy equivalence

$$h \circlearrowright (M, D_0) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (H(M), 0),$$

define

$$D'_r = \sum_{i_1+i_2+\cdots+i_k=r} \pi D_{i_1} h D_{i_2} h \cdots h D_{i_k} \iota \quad \text{for } r \geqslant 1.$$

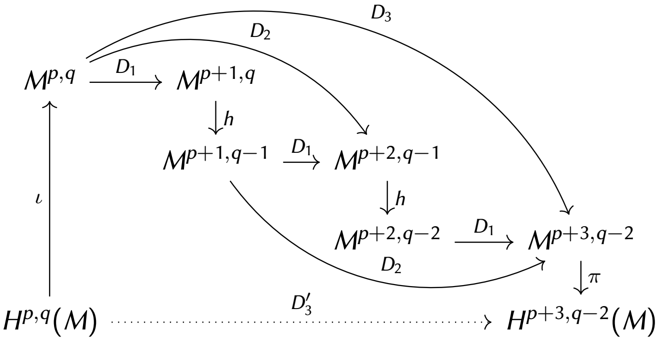
A special case of the HTT for multicomplexes

$(H(M), 0, D'_1, D'_2, \ldots)$ is a multicomplex.

Example — The differential D_3'

The transferred differential

$$D_3' = \pi(D_1 h D_1 h D_1 + D_1 h D_2 + D_2 h D_1 + D_3)\iota$$



Example - HTT applied to a double complex

If (M, D_0, D_1) is a double complex over a field, then the *multicomplex* $(H(M), 0, D'_1, D'_2, \ldots)$ is a lifted version of the usual spectral sequence associated to M:

Proposition

The map induced by D'_r on the E_r-page is precisely d_r (for all $r \geqslant 1$).

So far...

- Definition of a multicomplex.
- Some examples.
- The HTT allows us to transfer the higher differentials of M to $H(M)$.
- We recover the spectral sequence of a double complex via the HTT.

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- Definition of a multicomplex.
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- We recover the spectral sequence of a double complex via the HTT.

Next: The spectral sequence associated to a multicomplex.

A quick overview

$$(C^{\bullet,\bullet}, D_0, D_1) \longmapsto (\mathrm{Tot}(C)^{\bullet}, D) \longmapsto (E^{\bullet}(C), d_{\bullet})$$

$$\textbf{Double complexes} \xrightarrow{\mathrm{Tot}(-)} \textbf{Filtered Chain Complexes} \xrightarrow{E(-)} \textbf{Spectral sequences}$$

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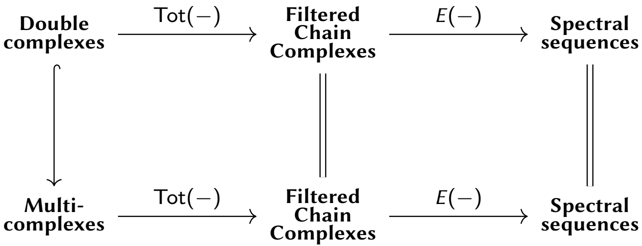
\end{array}$$

$$(M^{\bullet,\bullet}, \{D_r\}_{r\geqslant 0}) \longmapsto (\mathrm{Tot}(M)^{\bullet}, D') \longmapsto (E^{\bullet}(M), d_{\bullet})$$

A quick overview

Well-known:

How to compute the spectral sequence associated with a double complex over a field.



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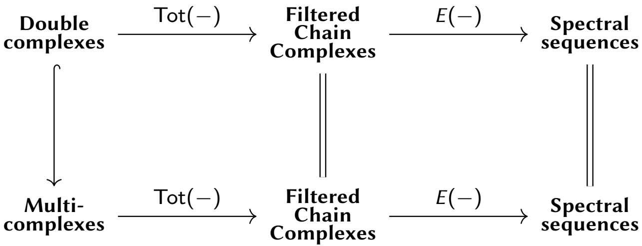

Goal:

Compute the spectral sequence associated with a multicomplex over a field.

The total complex

We define the *total complex* $\operatorname{Tot} M$ of a multicomplex $(M, D_\bullet)$ by

$$(\operatorname{Tot} M)^n = \bigoplus_{p+q=n} M^{p,q}$$

equipped with the differential

$$D = \sum_{r \geqslant 0} D_r \colon (\operatorname{Tot} M)^n \to (\operatorname{Tot} M)^{n+1}.$$

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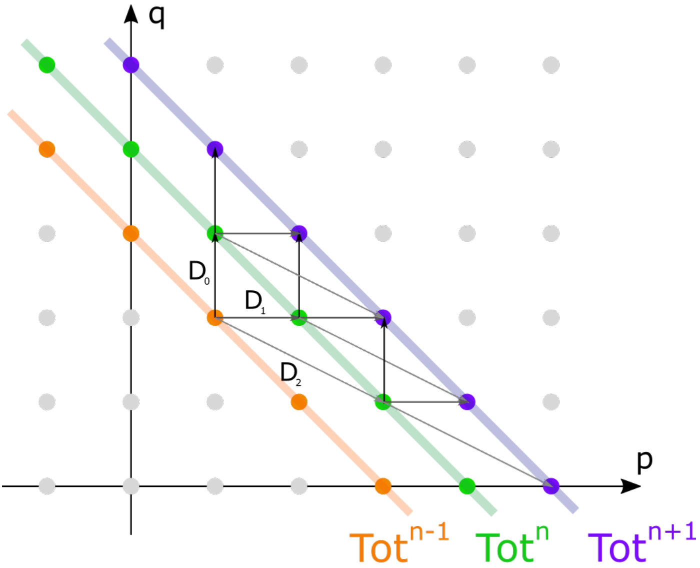
$$\sum_{p+q=n} D_p D_q = 0 \quad \text{for all } n \geqslant 0 \iff D^2 = 0.$$


q
p
D_0
D_1
D_2
Tot^{n-1}
Tot^{n}
Tot^{n+1}

Total complex — Example 1

$$M = \begin{array}{ccccc} \mathbb{K} & \xrightarrow{1} & \mathbb{K} & & \\ & & \uparrow{\scriptstyle 1} & & \\ & & \mathbb{K} & \xrightarrow{1} & \mathbb{K} \end{array}$$

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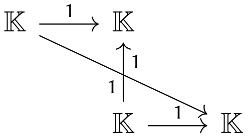
The differential $D = D_0 + D_1$ can be represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

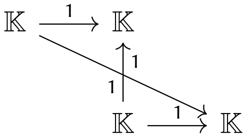
$H(\operatorname{Tot} M, D) = 0.$

Total complex — Example 2

$M =$



Total complex — Example 2

$M =$ 

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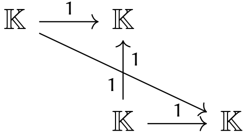

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The differential $D = D_0 + D_1 + D_2$ can be represented by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$H(\operatorname{Tot} M, D)$ is 1-dimensional in two degrees.

Filtration on total complex

We define a filtration F on $\operatorname{Tot} M$ by letting

$$F^a \operatorname{Tot} M^n = \bigoplus_{\substack{p+q=n \\ p \geqslant a}} M^{p,q} \qquad \left(\text{Note: } F^a \operatorname{Tot} M \geqslant F^{a+1} \operatorname{Tot} M\right).$$

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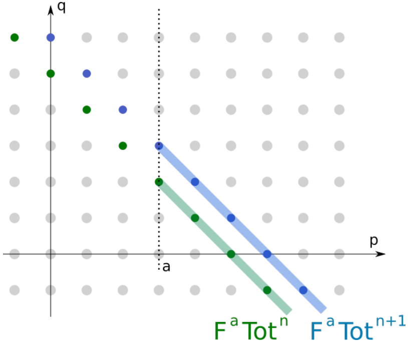
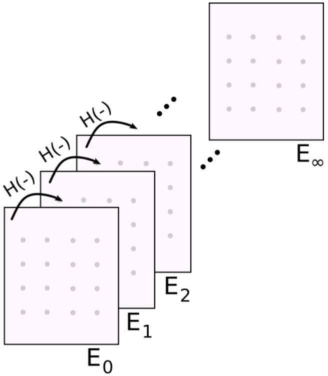


Spectral Sequences — Crash Course



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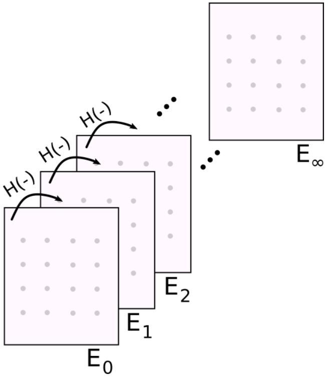

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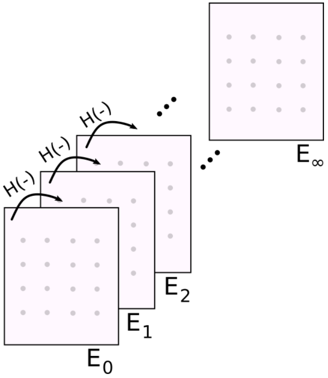

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- On each page there is a differential d_r.

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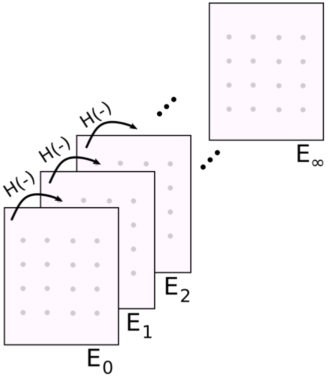

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- We pass to the next page by computing (co)homology:

$$E_{r+1} \cong H(E_r, d_r)$$

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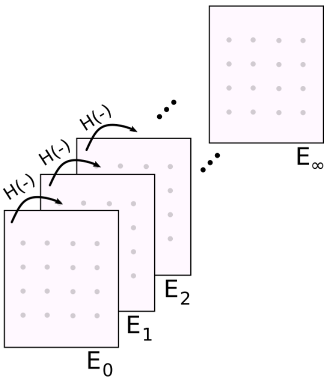

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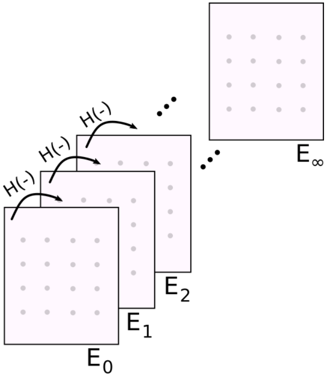

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- E_∞ is the "limit term".
- One way of getting a spectral sequence is to have a filtered chain complex (!)

Spectral sequence associated to a multicomplex

Theorem (Spanier95)

Let (C, F) be a filtered chain complex with F convergent and bounded below. Then there is a convergent spectral sequence with

$$E_0^{p,q} = F^p C^{p+q} / F^{p+1} C^{p+q}, \quad E_1^{p,q} = H^{p+q}(F^p C / F^{p+1} C)$$

and E_∞ is isomorphic to the associated graded of the induced filtration on $H(C)$.

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and E_∞ *is isomorphic to the associated graded of the induced filtration on* $H(C)$.

Takeaway

We get a spectral sequence from the filtration F on $\operatorname{Tot} M$. We call this *the spectral sequence associated with* M and denote it by $(E_\bullet(M), d_\bullet)$.

$$(M, D_\bullet) \mapsto (\operatorname{Tot} M, D, F) \mapsto (E_\bullet(M), d_\bullet)$$

The first pages

- $E_0 = M$ and $d_0 = D_0$

The first pages

- $E_0 = M$ and $d_0 = D_0$
- $E_1 = H(M)$ and $d_1 = H(D_1) = D'_1$

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- $E_2 = H(H(M), D'_1)$ and $d_2 = H(D'_2)$

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- $E_2 = H(H(M), D_1')$ and $d_2 = H(D_2')$

Warning

In general, d_r is *not* the map induced by D_r' whenever $r \geqslant 3$.

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Goal

Find a way to compute d_r in general.

Re-indexing the cohomology complex

We construct the multicomplex $({}^1M, {}^1D_\bullet)$ by setting

$$ {}^1M^{p,q} := H^{2p+q,-p}(M) \quad \text{and} \quad {}^1D_r := D'_{r+1}. $$

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Q: How is the spectral sequence associated with M related to the one associated with ${}^1M$?

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A: See next slide!

Main theorem

Theorem

The spectral sequence associated with ${}^1M$ *is a shifted version of the spectral sequence associated with* M *in the sense that*

$$E_r^{p,q}({}^1M) = E_{r+1}^{2p+q,-p}(M) \quad \text{and} \quad {}^1d_r = d_{r+1} \quad \text{for all} \quad r \geqslant 0.$$

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The spectral sequence associated with ${}^1M$ is a shifted version of the spectral sequence associated with M in the sense that

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Strategy to proving this:

Use an alternative description[1] of the spectral sequence in terms of witnessed cycles and boundaries. This gives a constructive proof.

[1] *On the spectral sequence associated to a multicomplex* (2019). Muriel Livernet, Sarah Whitehouse, Stephanie Ziegenhagen.

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- This way, we obtain a family of multicomplexes $({}^{s}M, {}^{s}D_{\bullet})$ where

$$ {}^{s}M^{p,q} := H^{2p+q,-p}({}^{s-1}M, {}^{s-1}D_0) \quad \text{and} \quad {}^{s}D_r := {}^{s-1}D'_{r+1}. $$

Corollary 1 — Computing differentials

Corollary

We have

$$E_r^{p,q}(M) = {}^r M^{p-rn,\, q+rn} \quad \textit{and} \quad d_r = {}^r D_0$$

for every $r \geqslant 1$ *where* $n = p + q$.

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Proof.

From theorem we have $E_r({}^1M) = E_{r+1}(M)$ for every $r \geqslant 0$.

$${}^rM = E_0({}^rM) = E_1({}^{r-1}M) = E_2({}^{r-2}M) = \cdots = E_{r-1}({}^1M) = E_r(M).$$

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Similarly, for the differentials, we get

$${}^rD_0 = {}^rd_0 = {}^{r-1}d_1 = {}^{r-2}d_2 = \cdots = {}^1d_{r-1} = d_r.$$

□

Relevant literature

- Multicomplexes and the associated spectral sequence:
  - *De Rham cohomology and homotopy Frobenius manifolds* — Vladimir Dotsenko, Sergey Shadrin, Bruno Vallette (2015)
  - *Multicomplexes and spectral sequences* — David E. Hurtubise (2009)
  - *On the spectral sequence associated to a multicomplex* — Muriel Livernet, Sarah Whitehouse, Stephanie Ziegenhagen (2019)
  - *Conditionally Convergent Spectral Sequences* — J. M. Boardman (1999)
- Other:
  - *Algebraic Operads* — Jean-Louis Loday, Bruno Vallette (2012)
  - *Model category structures on multicomplexes* — Xin Fu, Ai Guan, Muriel Livernet, Sarah Whitehouse (2021)

That's all!

Questions?

Extra slides

Corollary 2 — Degeneracy

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The spectral sequence associated with a multicomplex M degenerates at the k-th page if and only if ${}^{k}D_r = 0$ for all $r \geqslant 0$.

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The spectral sequence $E(M)$ degenerates at the k-th page $\iff$ $d_r = 0$ for all $r \geqslant k$ $\iff$ The spectral sequence $E({}^{k-1}M)$ degenerates at the first page

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The spectral sequence associated with a multicomplex M degenerates at the k-th page if and only if ${}^{k}D_r = 0$ for all $r \geqslant 0$.

Proof.

The spectral sequence $E(M)$ degenerates at the k-th page $\iff$ $d_r = 0$ for all $r \geqslant k$ $\iff$ The spectral sequence $E({}^{k-1}M)$ degenerates at the first page $\iff$ For all $r \geqslant 0$, we have $0 = {}^{k-1}D'_{r+1} = {}^{k}D_r$. □

Multicomplexes in the wild (not necessarily over a field)

- Double complexes, of course.
- The spectral sequence of a double complex.
- Resolutions for extensions of groups. (C.T.C. Wall, 1961)
- Resolutions of certain generalised Weyl algebras. (Liyu Liu, 2014 and 2017)

Remark: The limit term of $E_\bullet(M)$

We get an induced filtration on the total homology:

$$F^p H^\bullet(\operatorname{Tot} M) := \operatorname{im}(H^\bullet(F^p \operatorname{Tot} M) \to H^\bullet(\operatorname{Tot} M))$$

Convergence, written $E_\infty^{p,q} \Rightarrow H^{p+q}(\operatorname{Tot} M)$, means that

$$E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(\operatorname{Tot} M) = \frac{F^p H^{p+q}(\operatorname{Tot} M)}{F^{p+1} H^{p+q}(\operatorname{Tot} M)}.$$

Working over a field, we can read off the total homology:

$$H^n(\operatorname{Tot} M) = \bigoplus_{p+q=n} E_\infty^{p,q}.$$

Deformations of chain complexes

Multicomplexes can be identified with (polynomial/power series) curves in the "space" of chain complexes.

Homotopy algebras

"Multicomplexes are to double complexes as A_∞-algebras are to dg algebras."

- Double complexes are the dg-modules over the algebra D of dual numbers.
- Multicomplexes are the dg-modules over the Koszul resolution D_∞ of D.

Some open (?) questions

- Does every spectral sequence come from a multicomplex?
- Decomposition of multicomplexes into basic building blocks?
- What can we say over more general rings?
- What if we have a multiplicative structure around?

Why *spectral*?

Why *spectral*?

Nobody seems to really know:

"What is so "spectral" about spectral sequences?" — MathOverflow

Differentials exposed!

Let (π_0, ι_0, h_0) be a homotopy retract of M to ${}^1M = H(M)$ and (π_i, ι_i, h_i) be a homotopy retract of ${}^iM$ to ${}^{i+1}M$ for $i \geqslant 1$.

- $d_0 = D_0$
- $d_1 = \pi_0 D_1 \iota_0$
- $d_2 = {}^2D_0 = {}^1D'_1 = \pi_1\, {}^1D_1 \iota_1 = \pi_1 D'_2 \iota_1 = \pi_1 \pi_0 (D_1 h_0 D_1 + D_2) \iota_0 \iota_1$
- $$\begin{aligned} d_3 &= {}^3D_0 = {}^2D'_1 = \pi_2\, {}^2D_1 \iota_2 = \pi_2\, {}^1D'_2 \iota_2 = \pi_2 \pi_1 ({}^1D_1 h_1\, {}^1D_1 + {}^1D_2) \iota_1 \iota_2 \\ &= \pi_2 \pi_1 (D'_2 h_1 D'_2 + D'_3) \iota_1 \iota_2 \\ &= \pi_2 \pi_1 \pi_0 ((D_1 h_0 D_1 + D_2) \iota_0 h_1 \pi_0 (D_1 h_0 D_1 + D_2) \\ &\qquad + D_1 h_0 D_1 h_0 D_1 + D_1 h_0 D_2 + D_2 h_0 D_1 + D_3) \iota_0 \iota_1 \iota_2 \end{aligned}$$
- $d_4 =$ exercise.